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ON THE CIRCULAR FOOTING PLATES ON TWO-PARAMETERS FOUNDATION UNDER ARBITRARY LOADS

Loo Wen-da (卢文达) Wang Shu (王 蜀)

(*Shanghai Institute of Appl. Math. and Mech., Shanghai*)

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Abstract

Modelling soils by two-parameter foundation model, this paper calculates the distributions of displacements of circular footing plates on soils and reactions of soils under arbitrary loads using semi-analytical finite element method. And it improves V.Z.Vlazov's solution in the case of axisymmetry. The results agree well in comparison with those by F E M. At the same time, the boundary conditions of circular plates on soils are discussed.

I. Introduction

In recent years, the interaction problems between soils and structures are more and more important to people. In the analysis and design of raft and flat raft foundations, the bending properties of finite plates on soil medium are rather interesting. In the simplest case, soil medium is assumed as Winkle model[6], and then, circular plates on soil medium under axisymmetrical loads are considered. This was done by several persons in the 1950's. But such a model of soil medium is rather rough and impractical. Modelling soil medium as isotropic elastic half space (J. Boussinesq, 1885) [7], it is rather complicated in mathematics that circular plates on such a model under axisymmetrical loads are calculated. Circular plates on isotropic elastic half space under uniform load were calculated by B.N.Zemochkin (1939)[8], using Zemochkin method, and by A.G.Ishkova (1951)[9], using modified power series method. However, it has been found in practice that the surface displacements of soil far from the area loaded reduce more quickly than those predicted by elastic half space theory. Using the two-parameters model posed by P.L.Pasternak (1954)[10] to model soil medium, this paper calculates the circular footing plates on soils under arbitrary loads by semi-analytical F/E M. In the case of circular plate on two-parameters soil models under axisymmetrical loads, this paper improves Vlazov's results (1966)[6], and get the solutions of circular plates under arbitrary axisymmetrical loads. The solutions agree well in comparison with those by semi-analytical F E M.

II. Fundamental Equations and Boundary Conditions

The two-parameters soil model posed by P.L.Pasternak could be expressed as follows,

$$q(x,y)=kw(x,y)-G_p\nabla^2 w(x,y) \tag{2.1}$$

where $w(x,y)$ and $q(x,y)$ are displacement and pressure at the surface of soil, k, G_p are two parameters indicating features of soils, and they can be expressed as follows,

$$k=\int_0^H \frac{E_0}{(1-\nu_0^2)}\left[\frac{dh(z)}{dz}\right]^2 dz,\quad G_p=\frac{1}{2}\int_0^H \frac{E_0}{(1+\nu_0)}[h(z)]^2 dz \tag{2.2}$$

where $E_0=\dfrac{E_s}{(1-\nu_s)}$, $\nu_0=\dfrac{\nu_s}{(1-\nu_s)}$, E_s and ν_s are Young's modulus and Poisson's ratio, respectively; H, depth of soil; $h(z)$ describes the change of $w(x,y,z)$ along the depth direction z, i.e. $w(x,y,z)=w(x,y)h(z)$. Generally, $h(z)$ could be assumed as linear or exponential change, i.e.

$$h(z)=(1-\eta);\quad \text{or}\quad h(z)=\frac{\mathrm{sh}[\gamma(H-z)/R]}{\mathrm{sh}[\gamma H/R]} \tag{2.3}$$

where $\eta=z/H$, and γ , R, constants. It may be said that constant k is a dimension of deformation of soil medium under pressure, and G_p is a dimension of transmission of action to near elements, called ratio of transmission of load.

In the axisymmetrical case, the governing equation of circular plate on two-parameters soil medium is available as follows,

$$D\nabla^4 w(r)-G_p\nabla^2 w(r)+kw(r)=p(r) \tag{2.4}$$

where D is the bending rigidity of the plate; p is load.

In the case of general loads, the various strain energies are available by using polar coordinate.

a) strain energy in two-parameters soil medium,

$$U_s=\frac{1}{2}\int_0^{2\pi}\int_0^{R_s}\left\{k[w(r,\theta)]^2-\frac{1}{2}G_p w(r,\theta)\nabla^2 w(r,\theta)\right\}rdrd\theta \tag{2.5}$$

b) strain energy in circular plate (possibly nonuniform thickness),

$$U_p=\frac{1}{2}\int_0^{2\pi}\int_{r_p}^{R_p} D\left\{[\nabla^2 w(r,\theta)]^2-2(1-\nu)\left[\frac{\partial^2 w(r,\theta)}{\partial r^2}\left(\frac{1}{r}\frac{\partial w(r,\theta)}{\partial r}\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{r^2}\frac{\partial^2 w(r,\theta)}{\partial\theta^2}\right)-\left(\frac{1}{r}\frac{\partial^2 w(r,\theta)}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w(r,\theta)}{\partial\theta}\right)^2\right]\right\}rdrd\theta \tag{2.6}$$

c) work done by loads,

$$V=\int_0^{2\pi}\int_{r_p}^{R_p} q(r,\theta)w(r,\theta)rd\theta dr+\sum_i\int_0^{2\pi} p_i w_i rd\theta$$
$$-\sum_j\int_0^{2\pi} M_{r_j}(w_j)'_r rd\theta \tag{2.7}$$

On the circular plates on two-parameters soil medium, the boundary conditions could be divided into several cases as follows,

i) clamped at the edges of plate,

$$[w]_B=0,\quad \left[\frac{dw}{dr}\right]_B=0 \tag{2.8}$$

ii) simply supported at the edges of plate,

$$[w]_B=0,\quad \left[\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right]_B=M_0 \tag{2.9}$$

where M_0 is the moment density at the simply supported edge, and it is zero when there is no moment.

iii) In the case of the plate placed on soil medium freely and no constraints at its edges, we assume the displacement in plate as w_p, the displacement in soil medium out of the plate area as w_F, and R, the radius of plate at the boundary, then we have,

$$\left.\begin{aligned} & w_p(R)=w_F(R) \\ & \left[\frac{d^2w_p}{dr^2}+\frac{\nu}{r}\frac{dw_p}{dr}\right]_R=M_0 \\ & \left[-D\frac{d}{dr}\nabla^2 w_p+G_p\frac{dw_p}{dr}\right]_R=G_p\left[\frac{dw_F}{dr}\right]_R+T_0 \end{aligned}\right\} \tag{2.10}$$

where M_0 and T_0 are the moment and shear density at the free boundary, respectively.

III. Solution of F E M

Because the structure is axisymmetrical and loads are not, it is reasonable that semi-analytical finite element method is adopted. At first, we develop loads as Fourier series,

$$\left.\begin{aligned} q(r,\theta)&=\sum_{n=0}^{\infty}q_n(r)\cos n\theta \\ p(r,\theta)&=\sum_{n=0}^{\infty}p_n(r)\cos n\theta \\ M(r,\theta)&=\sum_{n=0}^{\infty}M_n(r)\cos n\theta \end{aligned}\right\} \tag{3.1}$$

and in the same way, displacement and forces in plate can also be developed into Fourier series,

$$\left.\begin{aligned} & w(r,\theta)=\sum_{n=0}^{\infty}w_n(r)\cos n\theta, \quad Q_r(r,\theta)=\sum_{n=0}^{\infty}Q_{p_n}(r)\cos n\theta \\ & M_r(r,\theta)=\sum_{n=0}^{\infty}M_{r_n}(r)\cos n\theta, \; M_\theta(r,\theta)=\sum_{n=0}^{\infty}M_{\theta_n}(r)\cos n\theta \end{aligned}\right\} \tag{3.2}$$

The total potential energy of the system could be obtained from (I) as follows,

$$\begin{aligned} U=U_p+U_s-V=&\frac{1}{2}\int_0^{2\pi}\int_{r_p}^{R_p}D\Big\{[\nabla^2 w(r,\theta)]^2 \\ &-2(1-\nu)\Big[\frac{\partial^2 w(r,\theta)}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w(r,\theta)}{\partial\theta^2} \\ &-\Big(\frac{1}{r}\frac{\partial^2 w(r,\theta)}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w(r,\theta)}{\partial\theta}\Big)^2\Big]\Big\}rdrd\theta+\frac{1}{2}\int_0^{2\pi}\int_0^{R_s}\Big\{k[w(r,\theta)]^2 \\ &-\frac{1}{2}G_p w(r,\theta)\nabla^2 w(r,\theta)\Big\}rdrd\theta-\int_0^{2\pi}\int_{r_p}^{R_p}q(r,\theta)w(r,\theta)rdrd\theta \\ &-\sum_i\int_0^{2\pi}p_i(r_i,\theta)w_i(r_i,\theta)r_i d\theta+\sum_i\int_0^{2\pi}M_i(r_i,\theta)[w(r_i,\theta)]'_{r_i}r_i d\theta \end{aligned} \tag{3.3}$$

As displacement and forces are all developed into Fourier series in θ -direction, we need only divide the circular plate into several concentric circular elements. Under the loads of each circumferential wave, the displacement in element is assumed as cubic polynomial in the radial direction, i.e.,

$$w(r,\theta)=(a_0+a_1r+a_2r^2+a_3r^3)\cos n\theta=w(r)\cos n\theta \tag{3.4}$$

so that the continuity of displacement w and angle deflection φ between elements is guaranteed.

Let

$$\{d\}=\{w_1,\varphi_1,w_2,\varphi_2\}^T \tag{3.5}$$

where (w_1,φ_1) and (w_2,φ_2) are the displacement and angle deflection of internal and external nodal lines of elements, and then,

$$\begin{Bmatrix} w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \end{Bmatrix}=\begin{bmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 0 & 1 & 2r_1 & 3r_1^2 \\ 1 & r_2 & r_2^2 & r_2^3 \\ 0 & 1 & 2r_2 & 3r_2^2 \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{3.6}$$

where r_1 and r_2 are the internal and external radius of elements, the above equation could be simply written as

$$\{d\}=[DA]\{a\} \tag{3.7}$$

so, $w(r)$ can be expressed as follows,

$$w(r)=\{1,r,r^2,r^3\}\{a\}=\{1,r,r^2,r^3\}[DA]^{-1}\{d\}=\{N\}\{d\} \tag{3.8}$$

Substituting the above equation into total potentail energy, equation (3.3), we obtain,

$$\begin{aligned} U=&\sum_{j=1}^{N}\int_0^{2\pi}\int_{r_j}^{r_{j+1}}\frac{1}{2}\{d\}^T[D]\{d\}rdrd\theta \\ &-\sum_{j=1}^{N}\left\{\int_0^{2\pi}\int_{r_j}^{r_{j+1}}\{d\}^T[N]^T\cos^2 n\theta q(r)rdrd\theta\right. \\ &+\sum_i\int_0^{2\pi}\{d\}^T[N]_i^T\cos^2 n\theta P_i(r_i)r_id\theta \\ &\left.-\sum_i\int_0^{2\pi}\{d\}^T([N]_i^T)'_i\cos^2 n\theta M_i(r_i)r_id\theta\right\} \end{aligned} \tag{3.9}$$

Integrating the above equation over every element, then we have,

$$U_{unit}=2^{-1}\{d\}^T[k]\{d\}-\{d\}^T\{p\} \tag{3.10}$$

With the static equilibrium condition $\delta U_{unit}=0$, we get,

$$[k]\{d\}=\{p\} \tag{3.11}$$

where $[k]$ is the stiffness matrix of element; $\{p\}$, load vector of element.

Because the local coordinates of elements and total coordinates are agreeable, the stiffness matrixes of elements can be straightly assembled into total stiffness matrixes without coordinate transformation.

IV. Axisymmetrical Analytical Solution

In (II), the governing equation of circular plate on two-parameters soil medium has been presented,

$$D\nabla^4 w(r) - G_p\nabla^2 w(r) + kw(r) = p(r) \tag{2.4$'$}$$

Introducing the dimensionless variables $\xi = r/L_0$, $L_0 = \sqrt[4]{D/k}$, we get,

$$\nabla_\xi^2\nabla_\xi^2 w - 2r_0^2\nabla_\xi^2 w + w = pL_0^4/D \tag{4.1}$$

where $r_0^2 = G_p L_0^2/2D$

The solution[5] of equation (4.1) can be divided into three parts,

$$w = w_1 + w_2 + w_p \tag{4.2}$$

where w_p is a special solution of equation (4.1), determined by the form of loads. And w_1 and w_2 are general solutions of the corresponding homogeneous equation of equation (4.1),

$$\left.\begin{aligned} w_1(\xi) &= B_1 J_0(\sqrt{a}\,\xi) + B_2 H_0^{(1)}(\sqrt{a}\,\xi) \\ w_2(\xi) &= B_3 J_0(\sqrt{\bar a}\,\xi) + B_4 H_0^{(2)}(\sqrt{\bar a}\,\xi) \end{aligned}\right\} \tag{4.3}$$

where $J_0(\sqrt{a}\,\xi)$ and $J_0(\sqrt{\bar a}\,\xi)$ are the first kind of Bessel functions of order zero with variables $\sqrt{a}\,\xi$ and $\sqrt{\bar a}\,\xi$; $H_0^{(1)}(\sqrt{a}\,\xi)$ and $H_0^{(2)}(\sqrt{\bar a}\,\xi)$ are the first kind and second kind of Hankle functions of order zero (the third kind of Bessel function), and,

$$a = -r_0^2 + i\sqrt{1-(r_0^2)^2}\,;\; \bar a = -r_0^2 - i\sqrt{1-(r_0^2)^2} \tag{4.4}$$

$J_0(\sqrt{a}\,\xi)$, $J_0(\sqrt{\bar a}\,\xi)$, $H_0^{(1)}(\sqrt{a}\,\xi)$ and $H_0^{(2)}(\sqrt{\bar a}\,\xi)$ are the complex functions, but w should be real function, so we express the result of w as follows,

$$w = c_1 u_0(\xi) + c_2 v_0(\xi) + c_3 f_0(\xi) + c_4 g_0(\xi) + w_p \tag{4.5}$$

where,

$$\left.\begin{aligned} u_0(\xi) &= \mathrm{Re}J_0(\sqrt{a}\,\xi) = (J_0\sqrt{a}\,\xi) + J_0(\sqrt{\bar a}\,\xi))/2 \\ v_0(\xi) &= \mathrm{Im}J_0(\sqrt{\bar a}\,\xi) = (J_0\sqrt{a}\,\xi) - J_0(\sqrt{\bar a}\,\xi))/2i \\ f_0(\xi) &= \mathrm{Re}H_0^{(1)}(\sqrt{a}\,\xi) = (H_0^{(1)}(\sqrt{a}\,\xi) + H_0^{(2)}(\sqrt{\bar a}\,\xi))/2 \\ g_0(\xi) &= \mathrm{Im}H_0^{(2)}(\sqrt{\bar a}\,\xi) = (H_0^{(1)}(\sqrt{a}\,\xi) - H_0^{(2)}(\sqrt{\bar a}\,\xi))/2i \end{aligned}\right\} \tag{4.6}$$

the internal forces can be deduced from the above equations as follows,

$$\frac{dw}{dr} = -\frac{1}{L_0}\left[c_1\theta_1(\xi) + c_2\theta_2(\xi) + c_3\theta_3(\xi) + c_4\theta_4(\xi) - \frac{dw_p}{d\xi}\right] \tag{4.7}$$

$$\begin{aligned} M_r = L_0^{-2}D\{&c_1[M_1(\xi) - (1-\nu)\overline{M}_1(\xi)] + c_2[M_2(\xi) \\ &- (1-\nu)\overline{M}_2(\xi)] + c_3[M_3(\xi) - (1-\nu)\overline{M}_3(\xi)] \\ &+ c_4[M_4(\xi) - (1-\nu)\overline{M}_4(\xi)] - \left[\nabla_\xi^2 - \frac{1-\nu}{\xi}\frac{d}{d\xi}\right]w_p\Big\} \end{aligned} \tag{4.8}$$

$$M_\theta = L_0^{-2}D\Big\{c_1[\nu M_1(\xi) + (1-\nu)\overline{M}_1(\xi)] + c_2[\nu M_2(\xi)$$

$$+(1-\nu)\overline{M}_2(\xi)]+c_3[\nu M_3(\xi)+(1-\nu)\overline{M}_3(\xi)]$$

$$+c_4[\nu M_4(\xi)+(1-\nu)\overline{M}_4(\xi)]-\left[\nu\nabla_\xi^2+\frac{1-\nu}{\xi}\frac{d}{d\xi}\right]w_p\Big\} \tag{4.9}$$

$$Q_r=-L_0^{-3}D[c_1Q_1(\xi)+c_2Q_2(\xi)+c_3Q_3(\xi)+c_4Q_4(\xi)+d\nabla_\xi^2 w_p/d\xi] \tag{4.10}$$

$$Q_c=kw+G_p(M_r+M_\theta)/[D(1+\nu)] \tag{4.11}$$

the following symbols are introduced in the above equations,

$$\left.\begin{array}{l}\theta_1(\xi)=u_1(\xi)\cos\varphi-v_1(\xi)\sin\varphi\ ,\ \ \theta_2(\xi)=u_1(\xi)\sin\varphi+v_1(\xi)\cos\varphi\\ \theta_3(\xi)=f_1(\xi)\cos\varphi-g_1(\xi)\sin\varphi\ ,\ \ \theta_4(\xi)=f_1(\xi)\sin\varphi+g_1(\xi)\cos\varphi\end{array}\right\} \tag{4.12}$$

$$\left.\begin{array}{l}M_1(\xi)=u_0(\xi)\cos2\varphi-v_0(\xi)\sin2\varphi,\ M_2(\xi)=u_0(\xi)\sin2\varphi+v_0(\xi)\cos2\varphi\\ M_3(\xi)=f_0(\xi)\cos2\varphi-g_0(\xi)\sin2\varphi,\ M_4(\xi)=f_0(\xi)\sin2\varphi+g_0(\xi)\cos2\varphi\end{array}\right\} \tag{4.13}$$

$$\left.\begin{array}{l}\overline{M}_1(\xi)=\xi^{-1}[u_1(\xi)\ \cos\varphi-v_1(\xi)\sin\varphi]\\ \overline{M}_2(\xi)=\xi^{-1}[u_1(\xi)\ \sin\varphi+v_1(\xi)\cos\varphi]\\ \overline{M}_3(\xi)=\xi^{-1}[f_1(\xi)\ \cos\varphi-g_1(\xi)\sin\varphi]\\ \overline{M}_4(\xi)=\xi^{-1}[f_1(\xi)\sin\varphi+g_1(\xi)\cos\varphi]\end{array}\right\} \tag{4.14}$$

$$\left.\begin{array}{l}Q_1(\xi)=u_1(\xi)\cos3\varphi-v_1(\xi)\sin3\varphi,\ \ Q_2(\xi)=u_1(\xi)\sin3\varphi+v_1(\xi)\cos3\varphi\\ Q_3(\xi)=f_1(\xi)\cos3\varphi-g_1(\xi)\sin3\varphi,\ Q_4(\xi)=f_1(\xi)\sin3\varphi+g_1(\xi)\cos3\varphi\end{array}\right\} \tag{4.15}$$

and

$$\varphi=2^{-1}\arg a,\ a=-r_0^2+i\sqrt{1-(r_0^2)^2} \tag{4.16}$$

$$\left.\begin{array}{l}u_1(\xi)=\mathrm{Re}J_1(\sqrt{a}\,\xi)=(J_1(\sqrt{a}\,\xi)+J_1(\sqrt{\bar a}\,\xi))/2\\ v_1(\xi)=\mathrm{Im}J_1(\sqrt{a}\,\xi)=(J_1(\sqrt{a}\,\xi)-J_1(\sqrt{\bar a}\,\xi))/2i\\ f_1(\xi)=\mathrm{Re}H_1^{(1)}(\sqrt{a}\,\xi)=(H_1^{(1)}(\sqrt{a}\,\xi)+H_1^{(2)}(\sqrt{\bar a}\,\xi))/2\\ g_1(\xi)=\mathrm{Im}H_1^{(2)}(\sqrt{a}\,\xi)=(H_1^{(1)}(\sqrt{a}\,\xi)-H_1^{(2)}(\sqrt{\bar a}\,\xi))/2i\end{array}\right\} \tag{4.17}$$

With the displacement w_p of soil medium out of circular plate, the governing equation is,

$$G_p\nabla_r^2 w_p-kw_p=0 \qquad (0\leqslant r\leqslant R_{in},\ R_{out}\leqslant r<\infty) \tag{4.18}$$

The solution can be expressed as follows,

$$w_p(r)=c_5I_0(ar)+c_6K_0(ar) \tag{4.19}$$

where c_5 and c_6 are undetermined constants; $a^2=k/G_p$; I_0 and K_0 are the first and second modified Bessel function of order zero.

Because soil medium is an unloaded aera, certain conditions must be satisfied by w_p.

i) in the area out of circular plate ($R_{out}\leqslant r<\infty$) , we have,

$$w_p(\infty)=0 \tag{4.20}$$

so it comes that $c_5=0$, i.e.,

$$w_p(r)=c_6K_0(ar) \tag{4.21}$$

ii) in the area within circular plate $0\leqslant r\leqslant R_{in}$,$w_p(0)$ should be a finite value, so we get $c_6=0$ (because $K_0(ar)$ is singular at point 0), i.e.,

$$w_F(r)=c_6 I_0(ar) \tag{4.22}$$

If there are line loads acting in circular plate, we can divide the circular plate into several ones at the line where loads act to be calculated respectively, and then, the undetermined coefficients can be determined by boundary conditions and continuous conditions between circular plates.

V. Examples and Conclusions

We consider the calculation of circular footing plate of hyperbolic cooling tower of 3500 M^2 (Fig.1). The soil foundation is a natural one with the parameters: elastic modulus $E_s=6$ kg/mm², Poisson's ratio $\nu_s=0.25$, depth $H_s=3$m. The internal radius of circular footing plate $r=34273$mm, external radius $R=39273$mm; with the uniform thickness, $h=1700$mm; with the variable thickness, $h_{max}=1000$mm, $h_{min}=460$mm; elastic modulus $E=2700$ Kg/mm², Poisson's ratio $\nu=0.3$. The loads are the discrete point loads transmitted from V-type columns of cooling tower to the circular footing plate, and they can be developed into concentrated line loads on circular footing plate through Fourier series. When only dead loads are considered, loads are axisymmetrical, and the analytical solutions and solution of FEM (uniform thickness) are shown in Fig.2. It can be seen from the figure that displacement w and reactions of soil Q calculated by the two methods are agreeable. When cooling tower is loaded simultaneously by dead load and wind load, loads are unaxisymmetrical (180° symmetrical), the reaction distribution of soil in radial and circumferential direction (variable thickness) are shown in Fig. 3. The following conclusions have been obtained through series calculations,

(i) the greater the elastic modulus of soil, the less the displacement of plate, but the greater the reactions of soil;

(ii) the deeper the depth of soil layer, the greater the displacement of plate but the less the reaction of soil;

(iii) the two-parameters model can model the static properties of soil well, and make the expressions in mathematics more convenient.

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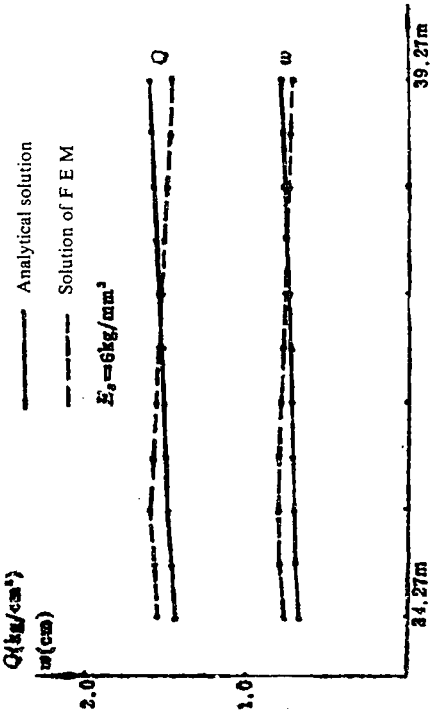

Fig. 2

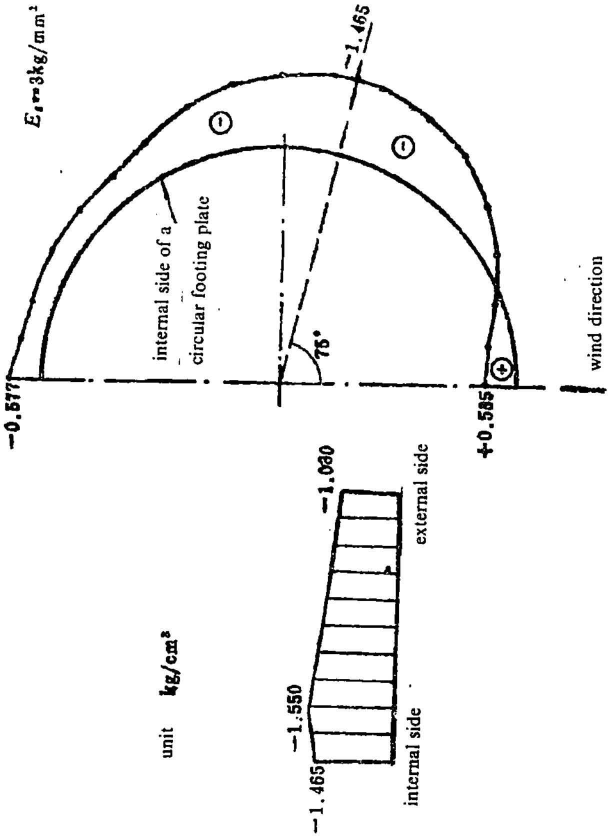

Fig. 3

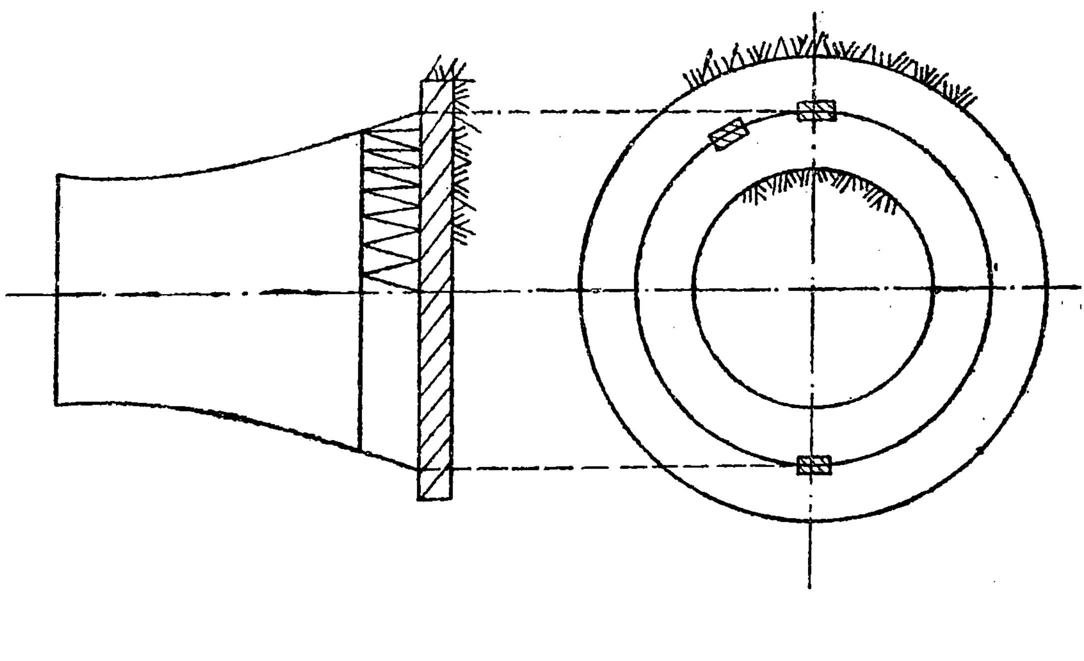

Fig. 1

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