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SOLUTION OF THE PLANE STRESS PROBLEMS OF STRAIN-HARDENING MATERIALS DESCRIBED BY POWER-LAW USING THE COMPLEX PSEUDO-STRESS FUNCTION

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Abstract

In the present paper, the compatibility equation for the plane stress problems of power-law materials is transformed into a biharmonic equation by introducing the so-called complex pseudo-stress function, which makes it possible to solve the elastic-plastic plane stress problems of strain hardening materials described by power-law using the complex variable function method like that in the linear elasticity theory. By using this general method, the close-formed analytical solutions for the stress, strain and displacement components of the plane stress problems of power-law materials is deduced in the paper, which can also be used to solve the elasto-plastic plane stress problems of strain-hardening materials other than that described by power-law. As an example, the problem of a power-law material infinite plate containing a circular hole under uniaxial tension is solved by using this method, the results of which are compared with those of a known asymptotic analytical solution obtained by the perturbation method.

Key words power-law materials, pseudo-stress function, plane stress problems, the complex variable function method, total deformation theory

I. Introduction

In general, metal materials used in engineering all have some strain hardening characteristics. Thus, it is of great practical importance to find a general solution to the plane problems of strain-hardening materials. However, owing to the mathematical difficulties arising from nonlinearity, few attempts have been successful so far in finding an analytical solution for such problems. In recent years, Chinese scholars have done some important work in this field. In reference [1], a general asymptotic analytical solution for the plane problems of strain-hardening materials was suggested by the perturbation method, assuming that the stress-strain relation of a material is expressed by a power series and a total deformation theory is applied to the problem. This solution is applicable to any plane stress or plane strain (incompressible) problem with known elastic solutions. Based on this solution, the asymptotic solution of a strain-hardening elasto-plastic plate containing an elliptical hole under equal-biaxial tension was obtained in reference [2], and the asymptotic analytical solution of the strain-hardenign elasto-plastic plate containing a circular hole under uniaxial tension was obtained in reference [3]. In addition, the study of stress concentration at circular holes in some strain-hardening material infinite plates was carried out in references [4] and [5] by using the finite element method, and the Fourier series method in conjunction with the finite differentiation method, respectively. However, each of those known solutions is either an asymptotic analytical solution or a numerical solution with a very complicated performing process, making it inconvenient to use directly in solving engineering problems. Recently, a complex variable function method for solving the plane strain problems of power-law materials by using complex pseudo-stress function was suggested in reference [6], where the stress concentration problems of a power-law infinite medium containing a single circular hole or rigid inclusion were solved by using the method. Unfortunately, the way of introducing a pseudo-stress function and the associated formulae presented in reference [6] cannot be directly applied to solve the plane stress problems without transformation because of the definite difference between the two expressions of effective stresses as well as the strain components. As for this situation, a nonlinear differential transformation different from that used in reference [6] is introduced in this paper, which transformed the nonlinear compatibility equation of the plane stress problems into a biharmonic equation. Thus it becomes possible to successfully overcome the mathematical difficulties met in solving the high-order nonlinear equation resulting from substituting the Airy stress function directly into the compatibility equation. Based on this mathematical simplification, the close-formed complex variable function analytical solutions for the stress, strain and displacement components of the plane stress problems of power-law materials are deduced and presented in the paper. As a result, an effective general analytical method for precisely solving the plane problems of strain-hardening materials described by power-law has been well developed by this paper in conjunction with reference [6] with the aid of the theories and methods established for the associated linear-elastic plane problems in reference [8]. As a proof example, the problem of a power-law material infinite plate containing a circular hole under uniaxial tension is solved by using this method, and its results are compared with those of a known asymptotic analytical solution. Furthermore, the stress concentration at the circular hole is also discussed by using the results.

II. Basic Theory and Formulae

Under the basic assumptions of small-scale deformation, no body force acting, volume incompressibility, proportional loading and the stress-strain relation described by power-law, the basic equations for the plane stress problem of a strain-hardening material abiding by a total deformation theory are as follows:

The **constitive equations** are

$$\sigma_i = A\varepsilon_i^n \tag{2.1}$$

$$\varepsilon_x = \frac{\varepsilon_i}{\sigma_i}\left(\sigma_x - \frac{1}{2}\sigma_y\right), \quad \varepsilon_y = \frac{\varepsilon_i}{\sigma_i}\left(\sigma_y - \frac{1}{2}\sigma_x\right), \quad \varepsilon_{xy} = \frac{3\varepsilon_i}{2\sigma_i}\tau_{xy} \tag{2.2}$$

The **equilibrium equations** are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \qquad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{2.3}$$

The **compatibility equation** is

$$\frac{\partial^2 \varepsilon_y}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial y^2}=2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{2.4}$$

where σ_i , ε_i in equations (2.1) and (2.2) are the effective stress and the effective strain, respectively, and A and n in equation (2.1) are material constants. The four groups of equations just listed above in conjunction with the given boundary conditions will form the plane stress boundary-value problem of a power-law material abiding by the total deformation theory described by equation (2.2).

Now introducing the Airy stress function $U(x, y)$, and letting

$$\sigma_x=\frac{\partial^2 U}{\partial y^2}, \quad \sigma_y=\frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy}=-\frac{\partial^2 U}{\partial x \partial y} \tag{2.5}$$

then the equilibrium equations (2.3) are satisfied. If equation (2.5) is substituted into equation (2.2), with the results of this performance immediately substituted into equation (2.4), we will obtain a high-order nonlinear equation for determining $U(x, y)$, which is almost impossible to be solved analytically. Thus, to overcome this difficulty, an appropriate kind of mathematical transformation has to be found, which should make the strain components expressed in terms of the Airy stress function $U(x, y)$ simple enough before they are substituted into the compatibility equation (2.5) to yield the final differential equation for determining the solution to the problem. In fact, the following method of introducing a pseudo-stress function can serve this purpose.

By introducing the complex conjugate variables $z=x+iy$ and $\bar{z}=x-iy$, the following differential relations are obtained:

$$\frac{\partial^2}{\partial y^2}=-\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)^2, \quad \frac{\partial^2}{\partial x^2}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)^2,$$

$$\frac{\partial^2}{\partial x \partial y}=i\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial \bar{z}^2}\right) \tag{2.6}$$

Substituting equation (2.6) into equation (2.5), we find

$$\sigma_x=-\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)^2 U, \quad \sigma_y=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)^2 U, \tag{2.7}$$

$$\tau_{xy}=-i\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial \bar{z}^2}\right)U$$

By using equation (2.7), the effective stress for a plane stress problem can be expressed as

$$\begin{aligned}\sigma_i&=[\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2]^{\frac{1}{2}}\\&=2\left[\left(\frac{\partial^2 U}{\partial z \partial \bar{z}}\right)^2+3\frac{\partial^2 U}{\partial z^2}\cdot\frac{\partial^2 U}{\partial \bar{z}^2}\right]^{\frac{1}{2}}\end{aligned} \tag{2.8}$$

Then, substituting equations (2.7) and (2.8) into equation (2.2), and taking equation (2.1) into consideration at the same time, we will get

$$\left.\begin{aligned}
\varepsilon_x&=\frac{1}{2}\left(\frac{2}{A}\right)^{\frac{1}{n}}\left[\left(\frac{\partial^2 U}{\partial z\partial\bar z}\right)^2+3\frac{\partial^2 U}{\partial z^2}\cdot\frac{\partial^2 U}{\partial\bar z^2}\right]^{\frac{1-n}{2n}}\left[-\frac{3}{2}\frac{\partial^2 U}{\partial z^2}\right.\\
&\quad\left.+\frac{\partial^2 U}{\partial z\partial\bar z}-\frac{3}{2}\frac{\partial^2 U}{\partial\bar z^2}\right]\\
\varepsilon_y&=\frac{1}{2}\left(\frac{2}{A}\right)^{\frac{1}{n}}\left[\left(\frac{\partial^2 U}{\partial z\partial\bar z}\right)^2+3\frac{\partial^2 U}{\partial z^2}\frac{\partial^2 U}{\partial\bar z^2}\right]^{\frac{1-n}{2n}}\left[\frac{3}{2}\frac{\partial^2 U}{\partial z^2}\right.\\
&\quad\left.+\frac{\partial^2 U}{\partial z\partial\bar z}+\frac{3}{2}\frac{\partial^2 U}{\partial\bar z^2}\right]\\
\varepsilon_{xy}&=-\frac{3i}{2}\,\frac{1}{2}\left(\frac{2}{A}\right)^{\frac{1}{n}}\left[\left(\frac{\partial^2 U}{\partial z\partial\bar z}\right)^2+3\frac{\partial^2 U}{\partial z^2}\,\frac{\partial^2 U}{\partial\bar z^2}\right]^{\frac{1-n}{2n}}\\
&\quad\cdot\left(\frac{\partial^2 U}{\partial z^2}-\frac{\partial^2 U}{\partial\bar z^2}\right)
\end{aligned}\right\}\tag{2.9}$$

It now becomes clear that the strain components expressed in terms of the Airy stress function $U(x, y)$ are in a very complicated form. To simplify them, we can perform an associated mathematical transformation as follows:

$$\left.\begin{aligned}
&\left[\left(\frac{\partial^2 U}{\partial z\partial\bar z}\right)^2+3\frac{\partial^2 U}{\partial z^2}\cdot\frac{\partial^2 U}{\partial\bar z^2}\right]^{\frac{1-n}{2n}}\frac{\partial^2 U}{\partial z^2}=\frac{\partial^2 A}{\partial z^2}\\
&\left[\left(\frac{\partial^2 U}{\partial z\partial\bar z}\right)^2+3\frac{\partial^2 U}{\partial z^2}\cdot\frac{\partial^2 U}{\partial\bar z^2}\right]^{\frac{1-n}{2n}}\frac{\partial^2 U}{\partial\bar z^2}=\frac{\partial^2 A}{\partial\bar z^2}\\
&\left[\left(\frac{\partial^2 U}{\partial z\partial\bar z}\right)^2+3\frac{\partial^2 U}{\partial z^2}\cdot\frac{\partial^2 U}{\partial\bar z^2}\right]^{\frac{1-n}{2n}}\frac{\partial^2 U}{\partial z\partial\bar z}=\frac{\partial^2 A}{\partial z\partial\bar z}
\end{aligned}\right\}\tag{2.10}$$

This is a nonlinear differential transformation. It is not hard to see that if $\partial^2U/\partial z^2$, $\partial^2U/\partial\bar z^2$ and $\partial^2U/\partial z\partial\bar z$ exist, then $\partial^2A/\partial z^2$, $\partial^2A/\partial\bar z^2$ and $\partial^2A/\partial z\partial\bar z$ also exist unless $\partial^2U/\partial z^2$ or $\partial^2U/\partial\bar z^2$ or $\partial^2U/\partial z\partial\bar z$ are singularities in a domain and on a boundary. By the way, it is especially noteworthy that equation (2.10) will yield $U\equiv A$ in the case of $n=1$, which is the case for a linear-elastic material.

Letting $k=k(A,n)=2^{-1}(2/A)^{\frac{1}{n}}$ and substituting equation (2.10) into equation (2.9), we will get

$$\left.\begin{aligned}
\varepsilon_x&=k\left[-\frac{3}{2}\frac{\partial^2 A}{\partial z^2}+\frac{\partial^2 A}{\partial z\partial\bar z}-\frac{3}{2}\frac{\partial^2 A}{\partial\bar z^2}\right]\\
\varepsilon_y&=k\left[\frac{3}{2}\frac{\partial^2 A}{\partial z^2}+\frac{\partial^2 A}{\partial z\partial\bar z}+\frac{3}{2}\frac{\partial^2 A}{\partial\bar z^2}\right]\\
\varepsilon_{xy}&=-\frac{3i}{2}k\left[\frac{\partial^2 A}{\partial z^2}-\frac{\partial^2 A}{\partial\bar z^2}\right]
\end{aligned}\right\}\tag{2.11}$$

Substituting equation (2.11) into the compatibility equation (2.4), and taking equation (2.6) into account at the same time, we now find the resulting equation will be

$$\frac{\partial^4 A}{\partial z^2\partial\bar z^2}=0\tag{2.12}$$

Equation (2.12) as shown is a complex biharmonic equation, therefore, Λ is a biharmonic function, and according to the Gousant formula the solution is given as

$$\Lambda=\frac{1}{2}[\bar{z}\varphi(z)+\chi(z)+z\overline{\varphi(z)}+\overline{\chi(z)}] \tag{2.13}$$

where $\varphi(z)$, $\chi(z)$ are holomorphic functions of z. Since the same kind of differential equation is satisfied by both the function Λ and the stress function U of plane problems in the linear elasticity theory, and the problem of seeking solutions for the plane stress problem of a power-law material has been finally transformed into a problem of determining the function Λ after performing the transformation shown in equation (2.10), we can safely call the very function Λ the pseudo-stress function similar to but different from that of the stress function U in the linear elasticity theory. According to the deducing process shown above, it is clear that the strain components expressed in terms of the pseudo-stress function in equation (2.11) are the strain components that satisfy the equilibrium equations. On the other hand, if equation (2.10) is substituted into equation (2.2), and equation (2.11) is used at the same time, we will easily find

$$\left.\begin{aligned}\sigma_x&=-\frac{\partial^2U}{\partial z^2}+2\frac{\partial^2U}{\partial z\partial\bar{z}}-\frac{\partial^2U}{\partial\bar{z}^2}\\ \sigma_y&=\frac{\partial^2U}{\partial z^2}+2\frac{\partial^2U}{\partial z\partial\bar{z}}+\frac{\partial^2U}{\partial\bar{z}^2}\\ \tau_{xy}&=-i\left(\frac{\partial^2U}{\partial z^2}-\frac{\partial^2U}{\partial\bar{z}^2}\right)\end{aligned}\right\} \tag{2.14}$$

This is just the same as that of equation (2.7); thus, the stress components associated with the constitive equations shown in equation (2.11) are the stress components that satisfy the equilibrium equations. The reversibility of the deducing process as shown above reveals that the method of introducing the pseudo-stress function just developed is feasible. Furthermore, we can derive the expressions of the second derivatives of the Airy stress function U with respect to the complex conjugates z and $\bar{z}$ which are expressed in terms of the known second derivatives of the pseudo-stress function with respect to the complex conjugates from equation (2.10). These expressions are

$$\left.\begin{aligned}\frac{\partial^2U}{\partial z^2}&=\left[\left(\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\right)^2+3\frac{\partial^2\Lambda}{\partial z^2}\frac{\partial^2\Lambda}{\partial\bar{z}^2}\right]^{\frac{n-1}{2}}\frac{\partial^2\Lambda}{\partial z^2}\\ \frac{\partial^2U}{\partial\bar{z}^2}&=\left[\left(\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\right)^2+3\frac{\partial^2\Lambda}{\partial z^2}\cdot\frac{\partial^2\Lambda}{\partial\bar{z}^2}\right]^{\frac{n-1}{2}}\frac{\partial^2\Lambda}{\partial\bar{z}^2}\\ \frac{\partial^2U}{\partial z\partial\bar{z}}&=\left[\left(\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\right)^2+3\frac{\partial^2\Lambda}{\partial z^2}\frac{\partial^2\Lambda}{\partial\bar{z}^2}\right]^{\frac{n-1}{2}}\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\end{aligned}\right\} \tag{2.15}$$

Therefore, from equations (2.14) and (2.15) we will get the expressions of stress components expressed in terms of the pseudo-stress function as follows:

$$\begin{aligned}\sigma_x+\sigma_y&=4\frac{\partial^2U}{\partial z\partial\bar{z}}=4\left[\left(\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\right)^2+3\frac{\partial^2\Lambda}{\partial z^2}\frac{\partial^2\Lambda}{\partial\bar{z}^2}\right]^{\frac{n-1}{2}}\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\\ \sigma_y-\sigma_x+2i\tau_{xy}&=4\frac{\partial^2U}{\partial z^2}=4\left[\left(\frac{\partial^2\Lambda}{\partial z\partial\bar{z}}\right)^2+3\frac{\partial^2\Lambda}{\partial z^2}\frac{\partial^2\Lambda}{\partial\bar{z}^2}\right]^{\frac{n-1}{2}}\frac{\partial^2\Lambda}{\partial z^2}\end{aligned}$$

Substituting equation (2.13) into this equation, we will find the expressions of stress components expressed by the complex variable functions. These expressions are

$$\sigma_x+\sigma_y=2^{3-n}[4\mathrm{Re}^2\varphi'(z)+3|\bar{z}\varphi''(z)+\chi''(z)|^2]^{\frac{n-1}{2}}\mathrm{Re}\varphi'(z)$$

$$\sigma_y-\sigma_x+2i\tau_{xy}=2^{2-n}[4\mathrm{Re}^2\varphi'(z)+3|\bar{z}\varphi''(z)+\chi''(z)|^2]^{\frac{n-1}{2}}[\bar{z}\varphi''(z)+\chi''(z)]$$

Let

$$F=F(z)=4\mathrm{Re}^2\varphi'(z)+3|\bar{z}\varphi''(z)+\chi''(z)|^2 \tag{2.16}$$

Then we will finally get

$$\left.\begin{aligned}&\sigma_x+\sigma_y=2^{3-n}F^{\frac{n-1}{2}}\mathrm{Re}\varphi'(z)\\&\sigma_y-\sigma_x+2i\tau_{xy}=2^{2-n}F^{\frac{n-1}{2}}[\bar{z}\varphi''(z)+\chi''(z)]\end{aligned}\right\} \tag{2.17}$$

In the plane polar coordinates, after performing some simple tansformation on equation (2.17), the stress components will be given as

$$\left.\begin{aligned}&\sigma_r+\sigma_\theta=\sigma_x+\sigma_y=2^{3-n}F^{\frac{n-1}{2}}\mathrm{Re}\varphi'(z)\\&\sigma_\theta-\sigma_r+2i\tau_{r\theta}=(\sigma_y-\sigma_x+2i\tau_{xy})\exp[2i\theta]\\&\qquad=2^{2-n}F^{\frac{n-1}{2}}[\bar{z}\varphi''(z)+\chi''(z)]\exp[2i\theta]\end{aligned}\right\} \tag{2.18}$$

As shown in equations (2.17) and (2.18), the stress components of plane stress problems of power-law materials expressed by the complex variable functions would have been identical with that of linear-elastic materials given in the linear elasticity theory if it had not been for the multiplier $2^{1-n}F^{\frac{n-1}{2}}$, which is related not only to the material constant n but to the position complex variable $z=x+iy$ as well.

If we substitute equation (2.13) into equation (2.11), we will alternatively obtain the expressions of strain components expressed by the complex variable functions as follows:

$$\left.\begin{aligned}&\varepsilon_x=k\left\{\mathrm{Re}\ \varphi'(z)-\frac{3}{2}\mathrm{Re}[\bar{z}\varphi''(z)+\chi''(z)]\right\}\\&\varepsilon_y=k\left\{\mathrm{Re}\ \varphi'(z)+\frac{3}{2}\mathrm{Re}[\bar{z}\varphi''(z)+\chi''(z)]\right\}\\&\varepsilon_{xy}=\frac{3}{2}k\mathrm{Im}[\bar{z}\varphi''(z)+\chi''(z)]\end{aligned}\right\} \tag{2.19}$$

Furthermore, by using the pseudo-stress function Λ, the expressions of displacement components expressed by the complex variable functions can also be obtained. Now let

$$\nabla^2\Lambda=\frac{\partial^2\Lambda}{\partial x^2}+\frac{\partial^2\Lambda}{\partial y^2}=\Omega$$

Then we will get

$$\frac{\partial^2 A}{\partial x^2}=\Omega-\frac{\partial^2 A}{\partial y^2},\quad \frac{\partial^2 A}{\partial y^2}=\Omega-\frac{\partial^2 A}{\partial x^2} \tag{2.20}$$

Obviously, Ω is a harmonic function. Now we take a holomorphic function $f(z)=\Omega+i\omega$, and let

$$\varphi(z)=p+iq=\frac{1}{4}\int f(z)dz \tag{2.21}$$

where $\varphi(z)$ is the holomorphic function originally defined in equation (2.13). Then, according to the Cauchy-Riemann conditions, there will be

$$\frac{\partial p}{\partial x}=\frac{\partial q}{\partial y}=\frac{1}{4}\Omega,\quad -\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}=\frac{1}{4}\omega \tag{2.22}$$

Now, rewriting equation (2.11) as

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}=\varepsilon_x=k\left[\frac{3}{2}\frac{\partial^2 A}{\partial y^2}-2\frac{\partial^2 A}{\partial z\partial \bar{z}}\right] \\ &\frac{\partial v}{\partial y}=\varepsilon_y=k\left[\frac{3}{2}\frac{\partial^2 A}{\partial x^2}-2\frac{\partial^2 A}{\partial z\partial \bar{z}}\right] \\ &\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\varepsilon_{xy}=k\left[-\frac{3}{2}\frac{\partial^2 A}{\partial x\partial y}\right] \end{aligned}\right\} \tag{2.23}$$

and taking the relation $\partial^2 A/\partial z\partial\bar{z}=[\varphi'(z)+\overline{\varphi'(z)}]/2=\partial p/\partial x=\Omega/4$ into consideration at the same time, and then substituting equation (2.19) into equation (2.23), using relations (2.21) and (2.22), we will finally get

$$\frac{\partial u}{\partial x}=k\left[4\frac{\partial p}{\partial x}-\frac{3}{2}\frac{\partial^2 A}{\partial x^2}\right],\quad \frac{\partial v}{\partial x}=k\left[4\frac{\partial q}{\partial y}-\frac{3}{2}\frac{\partial^2 A}{\partial y^2}\right]$$

After performing integration, the two equations yield

$$u=k\left[4p-\frac{3}{2}\frac{\partial A}{\partial x}\right]+f_1(y),\quad v=k\left[4q-\frac{3}{2}\frac{\partial A}{\partial y}\right]+f_2(x)$$

Then, substituting these equations into the third one of equation (2.23), and taking equation (2.22) into consideration at the same time, we will find

$$f_1'(y)++f_2'(x)=0$$

This equation implies $f_1(y)=ay+c_1$ and $f_2(x)=-ax+c_2$, which are related to rigid-body displacement that contributes to neither strain components nor stress components. Therefore, we can delete them from the related equations, and then we have

$$u+iv=k\left[4\varphi(z)-\frac{3}{2}\left(\frac{\partial A}{\partial x}+i\frac{\partial A}{\partial y}\right)\right]$$

Now referring to equation (2.13), we will finally get

$$u+iv=k\left[\frac{5}{2}\varphi(z)-\frac{3}{2}z\overline{\varphi'(z)}-\frac{3}{2}\overline{\chi'(z)}\right] \tag{2.24}$$

In the plane polar coordinates, equation (2.24) can be written as

$$u_r+iv_\theta=(u+iv)\exp[-i\theta]=k\Big[\frac{5}{2}\varphi(x)-\frac{3}{2}z\overline{\varphi'(z)}$$
$$-\frac{3}{2}\overline{\chi'(z)}\Big]\exp[-i\theta] \tag{2.25}$$

Up to now, we have obtained all expressions of stress, strain and displacement components of the plane stress problem of a power-law material by using the complex variable method with the aid of the pseudo-stress function that satisfies a biharmonic equation. As a result, the solution-seeking pattern for the plane stress problems of power-law materials has been developed by using the complex variable function method. For a particular problem, if the holomorphic functions $\varphi(z)$ and $\chi(z)$ can be determined by the given boundary conditions, then the solution for all the stress, strain and displacement components will easily be obtained by using the basic formulae presented in this section.

III. A Solution-Seeking Example

As an application example of the general method just developed above, now we consider the following problem:

As shown in Fig. 1, assuming that a power-law material infinite plate containing a circular hole of radius R is subjected to a uniaxial tension p in the x direction, to find the solution for the stress and displacement distributions of the plate.

The boundary conditions of this problem are

when $r=R$: $\sigma_r=0$ and $\tau_{r\theta}=0$ (a)

when $|z|\to\infty$: $\sigma_x=p$ and $\sigma_y=\tau_{xy}=0$ (b)

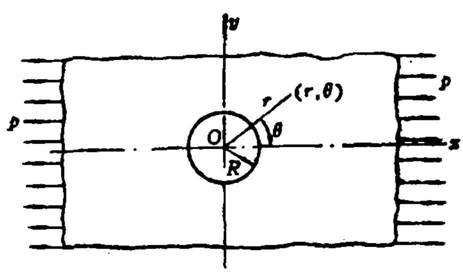

Fig. 1

Because this problem has the identical boundary conditios with its linear-elastic material counterpart, we can take the known solution of the latter as a model when we are about to choose the holomorphic functions expressed in the fashion of power series. To be specific, here we should take the same orders for the two power series expressing the two holomorphic functions which will define the pseudo-stress function Λ as those used to define the Airy stress function U in the associated linear-elastic problem, leaving only the constant coefficients in the power series to be determined here according to the boundary conditions given in both (a) and (b).The analysis shown above leads us to take

$$\varphi'(z)=a_0+\frac{a_2}{z^2},\quad \chi''(z)=a_0'+\frac{a_2'}{z^2}+\frac{a_4'}{z^4} \tag{c}$$

where a_0, a_2, a_0', a_2' and a_4' are the complex constant coefficients to be determined. Firstly, let us consider the boundary condition at infinity. Substituting equation (c) into equations (2.16) and (2.17), and letting $|z|\to\infty$, we will get

$$\left.\begin{aligned}2^{3-n}[4\mathrm{Re}^2a_0+3|a_0'|^2]\mathrm{Re}a_0=p\\ 2^{2-n}[4\mathrm{Re}^2a_0+3|a_0'|^2]a_0'=-p\end{aligned}\right\} \tag{d}$$

It is clear that from the second equation in (d) the coefficient a_0' must be a real constant. In addition, the coefficient a_0 can also be taken as a real costant if we learn from equations (2.17), (2.19) and (2.24) the fact that the imaginary part of a_0 is only related to rigid-body displacement. Thus, solving equation (d) will result in

$$a_0'=-\left(\frac{p}{2}\right)^{1/n},\quad a_0=\frac{1}{2}\left(\frac{p}{2}\right)^{1/n} \tag{e}$$

Now let us consider the boundary condition around the circular hole. Substituting the two equations into equation (2.18), we can get

$$2\sigma_r-2i\tau_{r\theta}=2^{2-n}F^{\frac{n-1}{2}}\{2\mathrm{Re}\varphi'(z)-[\bar{z}\varphi''(z)+\chi''(z)]\exp[2i\theta\} \tag{3.1}$$

Then, substituting equation (c) into equation (3.1), and rearranging it, we will find

$$2\sigma_r-2i\tau_{r\theta}=2^{2-n}F^{\frac{n-1}{2}}\Big[\Big(2a_0-\frac{a_2'}{r^2}\Big)+\Big(\frac{4a_2}{r^2}-\frac{a_4'}{r^4}-a_0'\Big)\cos 2\theta$$

$$+\Big(-a_0'-\frac{2a_2}{r^2}+\frac{a_4'}{r^4}\Big)\sin 2\theta\Big]$$

If we let $r=R$ and substitute equation (a) into this one, we will finally obtain a group of equations which are suitable for any θ value. These equations are

$$2a_0-a_2'/R^2=0$$

$$\frac{4a_2}{R^2}-\frac{a_4'}{R^4}-a_0'=0$$

$$-a_0'-\frac{2a_2}{R^2}+\frac{a_4'}{R^4}=0$$

Substituting equation (e) into this equation and then solving it, we will find

$$a_2'=\left(\frac{p}{2}\right)^{1/n}\cdot R^2,\quad a_2=-\left(\frac{p}{2}\right)^{1/n}R^2,\quad a_4'=-\left(\frac{p}{2}\right)^{1/n}\cdot 3R^4 \tag{f}$$

If we let $N=(p/2)^{1/n}$ in equations (e) and (f), then substitute the resulting coefficients into equation (c), and finally plug the results in equation (2.18), we will obtain the stress components as

$$\left.\begin{aligned}\sigma_\theta&=2^{1-n}F^{\frac{n-1}{2}}N\Big[1+\frac{R^2}{r^2}-\Big(1+\frac{3R^4}{r^4}\Big)\cos 2\theta\Big]\\ \sigma_r&=2^{1-n}F^{\frac{n-1}{2}}N\Big[1-\frac{R^2}{r^2}+\Big(1-\frac{R^2}{r^2}\Big)\Big(1-\frac{3R^2}{r^2}\Big)\cos 2\theta\Big]\\ \tau_{r\theta}&=2^{1-n}F^{\frac{n-1}{2}}N\Big(1+\frac{3R^2}{r^2}\Big)\Big(\frac{R^2}{r^2}-1\Big)\sin 2\theta\end{aligned}\right\} \tag{3.2}$$

where F can be determined according to equation (2.16) as

$$F=N^2\Big\{\Big(1-\frac{2R^2}{r^2}\cos 2\theta\Big)^2+3\Big[-1+\frac{R^2}{r^2}\cos 2\theta+\Big(\frac{2R^2}{r^2}-\frac{3R^4}{r^4}\Big)\cos 4\theta\Big]^2$$

$$+3\Big[\frac{R^2}{r^2}\sin 2\theta+\Big(\frac{2R^2}{r^2}+\frac{3R^4}{r^4}\Big)\sin 4\theta\Big]^2\Big\} \tag{3.3}$$

The equation (3.2) combined with that in (3.3) will yield the close-formed analytical expressions of stress components for the problem shown in Fig. 1. Obviously, if we let $n=1$, then we will get

$$\left.\begin{aligned}\sigma_\theta&=\frac{p}{2}\left(1+\frac{R^2}{r^2}\right)-\frac{p}{2}\left(1+\frac{3R^4}{r^4}\right)\cos 2\theta\\ \sigma_r&=\frac{p}{2}\left(1-\frac{R^2}{r^2}\right)+\frac{p}{2}\left(1-\frac{R^2}{r^2}\right)\left(1-\frac{3R^2}{r^2}\right)\cos 2\theta\\ \tau_{r\theta}&=\frac{p}{2}\left(1+\frac{3R^2}{r^2}\right)\left(\frac{R^2}{r^2}-1\right)\sin 2\theta\end{aligned}\right\} \tag{3.4}$$

This is the well-known solution of a linear-elastic material infinite plate containing a circular hole of radius R subjected to uniaxial tension p obtained by G. Kirsch.

Substituting the final complex holomorphic functions just stated and used to get equation (3.2) into equation (2.24), we can further obtain the displacement components as

$$\left.\begin{aligned}u_r&=kN\left[\left(\frac{r}{2}+\frac{3R^2}{2r}\right)+\left(\frac{4R^2}{r}+\frac{3r}{2}-\frac{3R^4}{2r^3}\right)\cos 2\theta\right]\\ v_\theta&=-kN\left[\left(\frac{R^2}{r}+\frac{3r}{2}+\frac{3R^4}{2r^3}\right)\sin 2\theta\right]\end{aligned}\right\} \tag{3.5}$$

To prove our solution, we take the known asymptotic analytical solution given in reference [3] by the perturbation method for the same kind of problem of a strain-hardening materical to make a comparison. To serve this purpose, we take the tangental stress at $\theta=\pi/2$ to compare. Thus, letting $\theta=\pi/2$in the first equation in (3.2) and equation (3.3), and substituting $N=(p/2)^{1/n}$, we will find

$$\sigma_\theta\Big|_{\theta=\frac{\pi}{2}}=2^{-n}p\left(2+\frac{R^2}{r^2}+\frac{3R^4}{r^4}\right)\left[4-\frac{2R^2}{r^2}+\frac{25R^4}{r^4}-\frac{18R^6}{r^6}+\frac{27R^8}{r^8}\right]^{\frac{n-1}{2}} \tag{3.6}$$

where n is the material constant to be determined. Since the following stress-strain relation

$$\varepsilon_i=\frac{\sigma_i}{200}+\frac{\sigma_i^3}{6600} \tag{h}$$

was used in one of the numerical examples in reference [3] to analyze the stress distribution of a strain-hardening material plate, we use the power-law relation to correlate the relation (h) in order to make a comparison with the solution given in reference [3] and to satisfy the requirement of our solution. The performance results in

$$\sigma_i=105.72\varepsilon_i^{0.899} \tag{k}$$

As shown in Fig. 2, the two curves given by (h) and (k) have a close correlation.

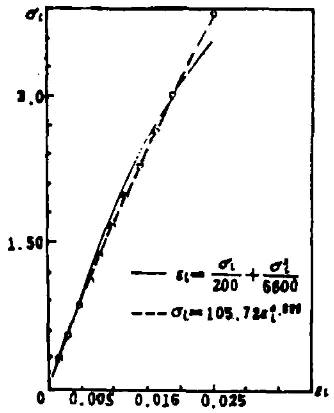

Fig. 2

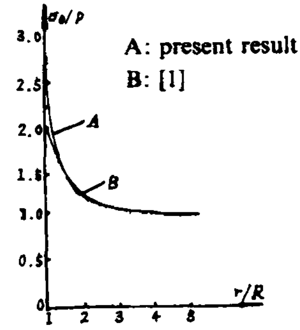

Fig. 3

The two tangental stress distributions at $\theta=\pi/2$ according to equation (3.6) of this paper and its counterpart in reference [3] are both drawn in Fig. 3, which would have made a very good contrast if it had not been for the errors near the hole. The asymptotic analytical solution itself is responsible for the errors. Now, let K_σ, K_ε be the stress and strain concentration factors of the power-law material plate, respectively, and K_t be the stress or strain concentration factor of its linear-elastic material counterpart. According to reference [7], there exists a relation $K_\sigma K_\varepsilon=K_t^2$ for any nonlinear material stress concentration problem, which is originally deduced from the the anti-plane strain cases subjected to shear loading. But as pointed out in reference [3], our solution for the plane stress problem of nonlinear power-law material also shows that the relation has some significant errors when it is applied to the plane problems subjected to tension loadin. If we let $r=R$, $\theta=\pi/2$ in equation (3.6), then we will get $K_\sigma=[\sigma_\theta]_{max}/p=3^n$, $K_\varepsilon=3$. Since the corresponding factor K_t is equal to 3, we finally get a relation of $K_\sigma K_\varepsilon=K_t^{n+1}$, which is applicable for any power-law material plate containing a circular hole under uniaxial tension. If we let $n=0.899$, we will find the stress concentration factor for our example to be $K_\sigma=2.685$.

IV. Conclusions

From the theoretical analysis and the particular application example shown above, we can safely conclude that the general method for solving the plane stress problem of power-law material by using the pseudo-stress function developed in the present paper is feasible. This is true for any nonlinear elastic problem or elasto-plastic problem without unloading. But for the elasto-plastic problem of power-law material with unloading, we can still obtain its solution by using this general method in conjunction with the linear elasticity theory. This is because there exists a linear-elastic relation between the effective stress and the effective strain during an unloading process. It is easy to see that this general method will also be effective for the stress and deformation analysis of a power-law material plate containing an arbitrary-shaped hole or inclusion or crack. In addition, since the power-law relation of $\sigma_i=A\varepsilon_i^n$ can be easily used to correlate the stress-strain relation of $\sigma_i=f(\varepsilon_i)$ for a given material, this general method can also be used to solve the elasto-plastic plane stress problem of a strain-hardening material other than that of a power-law material.

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References

[1] Gu Qiu-lin, A general asymptotic analytical solution of plane problem of elasto-plasticity with strain hardening, *Acta Mechanica Sinica*, **25**, 1 (1982). (in Chinese)

[2] Gu Qiu-lin,The asymptotic solution and experiment of a strain-hardening elasto-plastic plate containing an elliptical hole under tension, a paper contributed to the 9th U.S. National Congress of Applied Mechanics, Paper No. 149 (1982). (in Chinese)

[3] Dong Ya-min, Zhu Zhu-cheng and Gu Qiu-lin,The asymptotic analytical solution of the strain-hardening elasto-plastic plate containing a circular hole under simple tension, *Applied Mathematics and Mechanics*, **4**, 6 (1983).

[4] Marcal, P. V. and I. P. King, Elastic-plastic analysis of two-dimensional stress system by the finite element method, *Int. J. Mech. Sci.*, **9** (1967), 143.

[5] Huang, W. C., Theoretical study of stress concentration at circular holes and inclusions in strain-hardening materials, *Int. J. Solid. Struct.*, **8** (1967). 149.

[6] Lee Y. S., and H. Gong, Application of complex variables and pseudo-stress function to power-law materials and stress analysis of single rigid inclusion in power-law materials subjected to simple tension, 10/11 (1987). 669 – 694.

[7] Neuber, H., Theory of stress concentration for shear strained prismatical bodies in the arbitrary nonlinear stress-strain law, *J. Appl. Mech.*, **28** (1961).

[8] Muskhelishvili, H. I., *Some Basic Problems in the Mathematical Elasticity Theory* Science Press, Beijing (1958). (Chinese version)