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# INTERIOR LAYER BEHAVIOR OF BOUNDARY VALUE PROBLEMS FOR SECOND ORDER VECTOR EQUATION OF ELLIPTIC TYPE

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## Abstract

*In this paper, making use of the theory of partial differential inequalities, we will investigate the boundary value problems for a class of singularly perturbed second order vector elliptic equations, and obtain the existence and asymptotic estimation of solutions, involving the interior layer behavior, of the problems described above.*

**Key words** singular perturbation, vector elliptic equation, boundary value problem, upper and lower solutions, interior layer

## I. Existence of Solutions of Boundary Value Problems for Elliptic Equation

With the aid of the theory of partial differential inequalities, many authors have investigated the boundary value problems of singularly perturbed scalar elliptic equations[1-7]. However, the research for the vector elliptic boundary value problems is less to be seen. In this paper, making use of the method in [8-10], we will intense the results of scalar problems to the vector elliptic boundary value problems (BVPs) as follows:

$$\varepsilon\Delta y_i = f_i(x, y, \nabla y_i, \varepsilon), \quad x\in\Omega\subset R^m \tag{1.1}$$

$$y_i(x,\varepsilon) = g_i(x), \quad x\in\Gamma=\partial\Omega, \quad i\in I=\{1,2,\cdots,n\} \tag{1.2}$$

where $\varepsilon>0$ is a small parameter, $\Omega$ is a bounded open domain in $R^m$, and $\partial\Omega$ is a sufficiently smooth boundary.

For convenience, we assume that there exists a smooth function $F(x)$ such that $\Omega=\{x \text{ in } R^m;\ F(x)<0\}$ and $\partial\Omega=\{x \text{ in } R^m;\ F(x)=0\}$, where $\nabla F(x)\neq 0$, $x$ on $\Gamma$ and $\nabla F(x)$ is an outward normal to $\Gamma$. Let $J$ be a smooth closed curve properly contained in $\Omega$. We assume that $J$ is given by the equation $\Phi(x)=0$, where $\nabla\Phi(x)\neq 0$ along $J$. The curve $J$ divides $\Omega$ into two nonempty, disjoint open subsets $\Omega_1$ and $\Omega_2$ such that $\Omega=\Omega_1\cup\Omega_2\cup J$, and $\Omega_1=\{x \text{ in } \Omega;\ \Phi(x)<0\}$ and $\Omega_2=\{x \text{ in } \Omega;\ \Phi(x)>0\}$, and $\Omega_1$ be far away from $\Gamma$. For a given real number $a$ we set $J^a=\{x \text{ in } \Omega;\ |x-J|\leqslant a\}$. The set $J^a$ is a band of width $2a$ surrounding the curve $J$.

For the exposition, we shall take $\Omega$ to be a subset of $R^2$. In $J^a$, we use a local coordinate system $(r, z)$, where $r(x)=\pm|x-J|$, $r<0$ for $x$ in $\Omega_1$, $r>0$ for $x$ in $\Omega_2$, and where $z(x)$ is the arc length along $J$ from some reference point to the point on $J$ closest to $x$. Since $\Delta\Phi$ don't vanish along $J$. By continuity, for some $d>0$ the Jacobian of transformation $x\to(r, z)$ doesn't vanish in $J^d$. The coordinate trasformations are therefore bijective in $J^d$.

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In order to investigate the BVP (1.1), (1.2), we first consider the existence of solutions of the BVPs as follows:

$$\Delta y_i = f_i(x, y, \nabla y_i), \quad x \in \Omega \subset R^m \tag{1.3}$$

$$y_i(x) = g_i(x), \quad x \in \Gamma, \quad i \in I \tag{1.4}$$

**Definition** Let $\Omega$ be a bounded, open set in $R^m$. The functions $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ with $\alpha(x) \leqslant \beta(x)$ in $\Omega$ are called lower and upper solutions for the BVP (1.3), (1.4), respectively. if

(1) $\alpha, \beta \in C^2[\Omega_1 \cup \Omega_2, R^n]$, $D_l\alpha(x) \leqslant D_r\alpha(x)$ and $D_l\beta(x) \geqslant D_r\beta(x)$ on $J$, where $D_l$ and $D_r$ denote derivatives with respect to $r$ from the negative $r(\Omega_1)$ side of $J$ and the positive $r(\Omega_2)$ side of $J$, respectively;

(2) for each $i \in I$ and $\alpha_j \leqslant y_j \leqslant \beta_j$, $j \neq i$, the following inequalities are correct:

$$\Delta \alpha_i \geqslant f_i(x, y_1, \cdots, y_{i-1}, \alpha_i, y_{i+1}, \cdots, y_n, \nabla \alpha_i), \quad x \in \Omega_1 \cup \Omega_2$$

$$\Delta \beta_i \leqslant f_i(x, y_1, \cdots, y_{i-1}, \beta_i, y_{i+1}, \cdots, y_n, \nabla \beta_i), x \in \Omega_1 \cup \Omega_2$$

The principal tool we use in the study of the asymptotic behavior of solutions of the BVP (1.1), (1.2) is a theorem on partial differential inequalities as follows:

**Theorem 1** Suppose that in addition to the conditions on $\Omega$ described above, the following conditions hold:

(1) $f \in C^\mu[\bar{\Omega} \times R^n \times R^m, R^n]$, $g \in C^{1,\mu}[\partial\Omega, R^n]$, $0 < \mu < 1$, and $f_i(x, y, \nabla y_i)$ $(i \in I)$ are monotone nondecreasing in $y_i$;

(2) $f$ satisfies a Nagumo condition, that is, there exists an increasing function $\psi_i$: $R^+ \to R^+$ such that $|f_i(x, y, z)| \leqslant \psi_i(\|y\|)(1 + \|z\|^2)$ for all $(x, y, z) \in \bar{\Omega} \times R^n \times R^m$;

(3) there exist lower and upper solutions $\alpha$ and $\beta$ relative to (1.3) which satisfy $\alpha_i(x) \leqslant g_i(x) \leqslant \beta_i(x), x \in \partial\Omega, i \in I$.

Then there exists a solution $y = y(x) \in C^2[\bar{\Omega}, R^n]$ of the BVP (1.3), (1.4) such that

$$\alpha_i(x) \leqslant y_i(x) \leqslant \beta_i(x), \quad \alpha \in \bar{\Omega}, \quad i \in I \tag{1.5}$$

**Proof** Consider the modified nonlinear elliptic BVPs as follows:

$$\Delta y_i = F_i(x, y_i, \nabla y_i), \quad x \in \Omega \tag{1.6}$$

$$y_i(x) = g_i(x), \quad x \in \Gamma, \quad i \in I \tag{1.7}$$

where

$$F_i(x, y_i, \nabla y_i) = f_i(x, \mathcal{X}_1, \cdots, \mathcal{X}_{i-1}, y_i, \mathcal{X}_{i+1}, \cdots, \mathcal{X}_n, h(\nabla y_i)), i \in I, \tag{1.8}$$

and $\mathcal{X}_j (j \neq i) \in C^{1,\mu}[\bar{\Omega}, R]$, $\alpha_j(x) \leqslant \mathcal{X}_j \leqslant \beta_j(x)$, $x$ in $\bar{\Omega}$. $h \in C^1[R^m, R^m]$, $h(y) = y$ for $\|y\| \leqslant N$, $\|h(y)\| \leqslant \lambda\|y\|$ for $y$ in $R^m$, and $h(R^m)$, $h_y(R^m)$ are bounded, whene $h_y$ denotes the Jacobian matrix function of $h$, $\lambda > 1$, $N > \max\{\bar{N}, \max\limits_{\bar{\Omega}}\|\alpha_x(x)\|, \max\limits_{\bar{\Omega}}\|\beta_x(x)\|\}$, $\bar{N}$ being the Nagumo constant relative to $\alpha$, $\beta$ and $\Psi_i(i \in I)$. Similar to the proof of the corresponding theorems in [1] and [12], we may verify that there exists a solution $y = y(x) \in C^{2,\mu}[\bar{\Omega}, R^n]$ of the problem (1.6), (1.7) such that $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ and $\|y_x(x)\| \leqslant N$, $x$ in $\Omega$. Under the assumptions (1) – (3), making use of the monotone iterative technique which is used in [11], we may similarly prove that the BVP (1.3), (1.4) has a solution $y = y(x) \in C^2[\bar{\Omega}, R^n]$ satisfying $\alpha(x) \leqslant y(x) \leqslant \beta(x)$, $x \in \bar{\Omega}$. The details are omitted.

## II. Estimation of Solutions of Interior Behavior for Singularly Perturbed Problems

Next, we shall apply the previous Theorem and technique in [8]-[10] to investigate the existence and asymptotic estimation of the solutions involving the discontinuously reduced solutions of the BVP (1.1), (1.2). For brevity, we only discuss the case for $n=m=2$. Let $(u, v)$ be a solution pair of so-called reduced problems:

$$f_1(x,u,v,\nabla u,0)=0,\quad f_2(x,u,v,\nabla v,0)=0,\quad x\in\Omega \tag{2.1}$$

**Theorem 2** Suppose that in addition to the conditions on $\Omega$ described above, the following conditions hold:

(1) there exist solution pairs $(u_1, v_1)$ and $(u_2, v_2)$ of the reduced problem (2.1) such that $(u_1, v_1)\in C^2[\bar{\Omega}_1, R^2]$ and $(u_2,v_2)\in C^2[\bar{\Omega}_2,R^2]$, and $u_2(x)=g_1(x)$ and $v_2(x)=g_2(x)$ for $x$ on $\Gamma$, and $u_1(x)>u_2(x)$ and $v_1(x)<v_2(x)$ for $x$ on $J$;

(2) $f_i(x, y_1,y_2,p_1,p_2,\varepsilon)$ is assumed to be continuous in $x$ and $\varepsilon$, and continuously differentiable in $p_1$ and $p_2$ for all $(x, y_1,y_2,p_1,p_2,\varepsilon)\in\Theta$, where

$$\Theta=\{(x,y_1,y_2,p_1,p_2,\varepsilon): x\in\bar{\Omega}, |y_1-u(x)|\leqslant d_1(x),$$
$$|y_2-v(x)|\leqslant d_2(x), |p_i|<\infty, i=1,2, 0\leqslant\varepsilon\leqslant\varepsilon_0\},$$

$\varepsilon_0>0$ is a small constant, $d_i(x)$ $(i=1,2)$ are smooth positive functions: $d_1(x)=u_1(x)-u_2(x)+\delta$ on $J$, $d_1(x)=\delta$ in $\bar{\Omega}-J^\delta$; $d_2(x)=v_2(x)-v_1(x)+\delta$, $x\in J$, $d_2(x)=\delta$ in $\bar{\Omega}-J^\delta$, $J^\delta=\{x\in\Omega;\ |x-J|\leqslant\delta\}$;

(3) there exist positive constants $k_i, m_i$ $(i=1,2)$ such that

$$f_{iy_i}\geqslant k_i,\quad i=1,2,\quad |f_{1y_2}|\leqslant m_1,\quad |f_{2y_1}|\leqslant m_2 \text{ in } \Theta$$

(4)
$$(f_{1p_1}, f_{1p_2})(x,u_1,v_1,\nabla u_1,\varepsilon)\cdot\nabla\Phi>0,\quad x\in J^\delta\cap\bar{\Omega}_1$$
$$(f_{1p_1}, f_{1p_2})(x,u_2,v_2,\nabla u_2,\varepsilon)\cdot\nabla\Phi<0,\quad x\in J^\delta\cap\bar{\Omega}_2$$
$$(f_{2p_1}, f_{2p_2})(x,u_1,v_1,\nabla v_1,\varepsilon)\cdot\nabla\Phi>0,\quad x\in J^\delta\cap\bar{\Omega}_1$$
$$(f_{2p_1}, f_{2p_2})(x,u_2,v_2,\nabla v_2,\varepsilon)\cdot\nabla\Phi<0,\quad x\in J^\delta\cap\bar{\Omega}_2,$$

(5) the vector fields $(f_{1p_1}, f_{1p_2})(x, u_1, v_1, \nabla u_1, \varepsilon)$ and $(f_{2p_1},f_{2p_2})(x,u_1,v_1,\nabla v_1,\varepsilon)$ do not vanish and change symbols in $\Omega_1$, and $(f_{1p_1}, f_{1p_2})(x, u_2, v_2, \nabla u_2, \varepsilon)$ and $(f_{2p_1},f_{2p_2})(x,u_2,v_2,\nabla v_2,\varepsilon)$ do not vanish and change symbols in $\Omega_2$;

(6) $f_i$ $(i=1,2)$ satisfy Nagumo conditions:

(7) $f_1(x,u_i,v_i,\nabla u_i,\varepsilon)=O(\varepsilon)$ for $x\in\Omega_1$, $f_1(x,u_i,v_i,\nabla u_i,\varepsilon)=O(\varepsilon)$ for $x\in\Omega_2$

$f_2(x,u_i,v_i,\nabla u_i,\varepsilon)=O(\varepsilon)$ for $x\in\Omega_1$, $f_2(x,u_i,v_i,\nabla u_i,\varepsilon)=O(\varepsilon)$ for $x\in\Omega_2$

Then for $\varepsilon$ sufficiently small there exists a solution $y=y(x,\varepsilon)=(y_1(x,\varepsilon), y_2(x,\varepsilon))$ of the BVP (1.1), (1.2) such that

$$y_1(x,\varepsilon)=\begin{cases} u_1(x)+\bar{\xi}_1(x,\varepsilon)+O(\varepsilon) & (x\in\Omega_1)\\ u_2(x)+\bar{\xi}_2(x,\varepsilon)+O(\varepsilon) & (x\in\Omega_2)\end{cases} \tag{2.2}$$

$$y_2(x,\varepsilon)=\begin{cases} v_1(x)+\chi_1(x,\varepsilon)+O(\varepsilon) & (x\in\Omega_1)\\ v_2(x)+\chi_2(x,\varepsilon)+O(\varepsilon) & (x\in\Omega_2)\end{cases} \tag{2.3}$$

where $\bar{\xi}_i$ and $\bar{\chi}_i$ ($i=1, 2$) are interior layer corrections to be determined.

**Proof** The main idea of the proof is to construct the lower and upper solutions as in Theorem 1. We assume that for $\delta$ properly small $J^\delta$ is completely contained in $\Omega$. We know from the assumption (5) that there exists a constant vector $\nu=(\nu_1,\nu_2)$ such that

$$\begin{aligned}&(f_{1p_1},f_{1p_2})(x,u,v,\nabla u,\varepsilon)\cdot\nu>1\,(<-1),\quad x\in\Omega_1(\Omega_2)\\&(f_{2p_1},f_{2p_2})(x,u,v,\nabla v,\varepsilon)\cdot\nu>1\,(<-1).\quad x\in\Omega_1(\Omega_2)\end{aligned}\tag{2.4}$$

For notational ease, we set $h(x)=\nu_1x_1+\nu_2x_2$, so that $\nu=\nabla h$.

For $y_1$ we define $\alpha_1$ and $\beta_1$ as follows:

$$\alpha_1(x,\varepsilon)=\begin{cases}u_2(x)-\bar{\psi}_1(x,\varepsilon)-\varepsilon\gamma\exp[\lambda h(x)], & x\in\Omega_2\\ u_1(x)-\bar{\xi}_1(x,\varepsilon)-\bar{\psi}_2(x,\varepsilon)-\varepsilon\gamma\exp[\lambda h(x)], & x\in\bar{\Omega}_1\end{cases}$$

$$\beta_1(x,\varepsilon)=\begin{cases}u_2(x)+\bar{\xi}_2(x,\varepsilon)+\bar{\psi}_1(x,\varepsilon)+\varepsilon\gamma\exp[\lambda h(x)], & x\in\Omega_2\\ u_1(x)+\bar{\psi}_2(x,\varepsilon)+\varepsilon\gamma\exp[\lambda h(x)], & x\in\bar{\Omega}_1\end{cases}$$

where $\gamma$ is a positive constant to be determined, and $\lambda>\max\{m_1,m_2\}$, and $\bar{\psi}_1(x,\varepsilon)=\bar{\omega}(r)\psi_1(r,\varepsilon)$, $x\in J^\delta$ for $0\leqslant r\leqslant\delta$, $\bar{\psi}_1(x,\varepsilon)=0$, $x\in\Omega_2-J^\delta$, and $\bar{\psi}_2(x,\varepsilon)=\bar{\omega}(r)\psi_2(x,\varepsilon)$, $x\in J^\delta$ for $-\delta\leqslant r\leqslant0$, $\bar{\psi}_2(x,\varepsilon)=0$, $x\in\Omega_1-J^\delta$, where $\bar{\omega}(r)$ is a $C^2$-cut-off function such that $\bar{\omega}(r)=1$ for $|r|\leqslant3\delta/4$, $\bar{\omega}(r)=0$ for $|r|\geqslant4\delta/5$ and $0\leqslant\bar{\omega}\leqslant1$ for $|r|\leqslant\delta$, and where $\psi_i(i=1,2)$ are positive functions to be determined which satisfy $\psi_i=O(\varepsilon)$ and $\psi_{ir}=O(1)>0$, $\bar{\xi}_1(x,\varepsilon)=\omega(r)\zeta_1(r,\varepsilon)$, $x\in J^\delta$ for $0\leqslant r\leqslant\delta$, and $\bar{\xi}_1(x,\varepsilon)=0$, $x\in\Omega_2-J^\delta$, and $\bar{\xi}_2(x,\varepsilon)=\omega(r)\zeta_2(r,\varepsilon)$, $x\in J^\delta$ for $-\delta\leqslant r\leqslant0$ and $\bar{\xi}_2(x,\varepsilon)=0$, $x\in\Omega_1-J^\delta$, where $\bar{\omega}(r)$ is a $C^2$-cut-off function such that $\omega(r)=1$ for $|r|\leqslant\delta/2$, $\omega(r)=0$ for $|r|\geqslant3\delta/4$ and $0\leqslant\omega\leqslant1$ for $|r|\leqslant\delta$, and where $\zeta_i(i=1,2)$ satisfy

$$\begin{cases}\varepsilon\zeta_{1rr}-l_1\zeta_{1r}=-\kappa_1\zeta_{1r},\\ \zeta_1(0,\varepsilon)=\max(u_1(0,2)-u_2(0,2)),\zeta_1>0\\ \zeta_{1r}(0,\varepsilon)>0,\lim\limits_{\varepsilon\to0}\zeta_{1r}(0,\varepsilon)=\infty\\ \zeta_1\to0\text{ exponentially as }\varepsilon\to0\text{ for }r<0\end{cases}\tag{2.5}$$

$$\begin{cases}\varepsilon\zeta_{2rr}+l_2\zeta_{2r}=\kappa_2\zeta_{2r}\\ \zeta_2(0,\varepsilon)=\max(u_1(0,2)-u_2(0,2)),\ \zeta_2>0\\ \zeta_{2r}<0,\ \lim\limits_{\varepsilon\to0}\zeta_{2r}(0,\varepsilon)=-\infty\\ \zeta_2\to0\text{ exponentially as }\varepsilon\to0\text{ for }r>0\end{cases}\tag{2.6}$$

where $l_i(i=1,2)$ are positive constants which are selected from (4), (5) and satisfy $(f_{1p_1},f_{1p_2})(x,u,v,\nabla u,\varepsilon)\cdot(\partial t/\partial x_1,\partial t/\partial x_2)\geqslant l_1(\leqslant-l_2)$, $x\in J^\delta\cap\bar{\Omega}_1(J^\delta\cap\bar{\Omega}_2)$, and $(f_{2p_1},f_{2p_2})(x,u,v,\nabla v,\varepsilon)\cdot(\partial t/\partial x_1,\partial t/\partial x_2)\geqslant l_1(\leqslant-l_2)$, $x\in J^\delta\cap\Omega_1(J^\delta\cap\Omega_2)$, $\kappa_i(i=1,2)$ are small positive constants that is less than $l_i$.

It is clear from the construction of $\alpha_1$ and $\beta_1$ that $\alpha_1(x,\varepsilon)\leqslant\beta_1(x,\varepsilon)$ in $\bar{\Omega}$, and that $\alpha_1(x)\leqslant g_1(x)\leqslant\beta_1(x,\varepsilon)$ on $\Gamma$. The function $\alpha_1$ and $\beta_1$ are clearly not of class $C^2$ on all of $\Omega$ since these functions have discontinuous gradients across the curve $J$. However, along $J$ we have

$$D_l\alpha_1(0,z,\varepsilon)=D_lu_1(0,z)-\zeta_{1r}(0,\varepsilon)+O(\varepsilon)$$
$$<D_ru_2(0,z)+O(\varepsilon)=D_r\alpha_1(0,z,\varepsilon) \tag{2.7}$$

and

$$D_l\beta_1(0,z,\varepsilon)=D_lu_1(0,z)+O(\varepsilon)$$
$$>D_ru_2(0,z)+\zeta_{2r}(0,\varepsilon)+O(\varepsilon)=D_r\beta_1(0,z,\varepsilon) \tag{2.8}$$

for $\varepsilon$ sufficiently small. Thus $\alpha_1$ and $\beta_1$ satisfy the appropriate differential inequalities along $J$.

Now, we verify that $\alpha_1$ and $\beta_1$ are lower and upper solutions of the problem (1.1) in $\Omega_1\cup\Omega_2$. Similar to the proof in Kelley[8], we suppose that $y_2$ satisfies

$$v_2(x)-\chi(x,\varepsilon)-\varepsilon\gamma\exp[\lambda h(x)]-O(\varepsilon)\leqslant y_2$$
$$\leqslant v_2(x)+\varepsilon\gamma\exp[\lambda h(x)]+O(\varepsilon),\quad x\in\Omega_2 \tag{2.9}$$
$$v_1(x)-\varepsilon\gamma\exp[\lambda h(x)]-O(\varepsilon)\leqslant y_2$$
$$\leqslant v_1(x)+\overline{\chi}_2(x,\varepsilon)+\varepsilon\gamma\exp[\lambda h(x)]+O(\varepsilon),\quad x\in\Omega_1 \tag{2.10}$$

where $O(\varepsilon)(>0)$ is independent of $\lambda$, $\gamma$ and $\chi_i$ satisfies the equations which $\zeta_i$ satifies instead of $\chi_1(0,\varepsilon)=\max(v_2(0,z)-v_1(0,z))$ in (2.5), and $\chi_2(0,\varepsilon)=\max(v_2(0,z)-v_1(0,z))$ in (2.6).

Now, let us first prove that $\alpha_1$ is a lower solution in $\Omega-J^{3\delta/4}$. In this case, $\psi_i=\zeta_i=\chi_i=0\,(i=1,2)$, that is

$$\alpha_1(x,\varepsilon)=u(x)-\varepsilon\gamma\exp[\lambda h(x)]\qquad(x\in\Omega-J^{3\delta/4}) \tag{2.11}$$
$$v(x)-\varepsilon\gamma\exp[\lambda h(x)]-O(\varepsilon)\leqslant y_2\leqslant v(x)+\varepsilon\gamma\exp[\lambda h(x)]+O(\varepsilon),$$
$$x\in\Omega-J^{3\delta/4} \tag{2.12}$$

Since $\Delta u=[r^2_{x_1}+r^2_{x_2}]u_{rr}+[z^2_{x_1}+z^2_{x_2}]u_{zz}+[r_{x_1x_1}+r_{x_2x_2}]u_r+[z_{x_1x_1}+z_{x_2x_2}]u_z$ and $r_{x_1}+r_{x_2}=1$ and $\nabla r\cdot\nabla z=0$. Let $\Lambda=(r_{x_1x_1}+r_{x_2x_2})$, by the Mean Value Theorem we have

$$\varepsilon\Delta\alpha_1-f_1(x,\alpha_1,y_2,\nabla\alpha_1,\varepsilon)$$
$$\geqslant\varepsilon\Delta u-\varepsilon^2\gamma\Delta\exp[\lambda h(x)]-f_1(x,u,v,\nabla u,\varepsilon)+k_1\varepsilon\gamma\exp[\lambda h(x)]$$
$$-m_1\varepsilon\gamma\exp[\lambda h(x)]+\varepsilon\gamma\lambda\exp[\lambda h(x)]+O(\varepsilon)$$
$$\geqslant\varepsilon\gamma(k_1+\lambda-m_1)\exp[\lambda h(x)]+O(\varepsilon)+O(\varepsilon^2) \tag{2.13}$$

where $O(\varepsilon)$ is independent of $\lambda$, $\gamma$. Hence for $\lambda>m_1-k_1$, and $\gamma$ large enough and $\varepsilon$ sufficiently small we know that (2.13) is larger than or equal to zero. It remains to verify that $\alpha_1$ and $\beta_1$ satisfy the appropriate differential inequalities in $J^{3\delta/4}-J$.

Next, consider the function $\alpha_1$ in $J^{\delta/2}\cap\Omega_2$, we have

$$\varepsilon\Delta\alpha_1-f_1(x,\alpha_1,y_2,\nabla\alpha_1,\varepsilon)$$
$$=\varepsilon\Delta u_2-\varepsilon\psi_{1rr}-\varepsilon\Lambda\psi_{1r}-\varepsilon^2\gamma\Delta\exp[\lambda h(x)]-f_1(x,u_2,v_2,\nabla u_2,\varepsilon)$$
$$+k_1(\psi_1+\varepsilon\gamma\exp[\lambda h(x)])-m_1(\chi_1+\varepsilon\gamma\exp[\lambda h(x)])$$
$$+l_2\psi_{1r}+\varepsilon\gamma\lambda\exp[\lambda h(x)]+O(\varepsilon)$$
$$\geqslant-(\varepsilon\psi_{1rr}-l_2\psi_{1r}+m_1\chi_1)+\varepsilon\gamma(k_1+\lambda-m_1)\exp[\lambda h(x)]+O(\varepsilon)+O(\varepsilon^2) \tag{2.14}$$

We define $\psi_1$ as follows:

$$\psi_1(r,\varepsilon)=\varepsilon\int_0^{r/\varepsilon}\exp[l_2\tau]\int_\tau^\infty\exp[-l_2s]m_1\chi_1(s,\varepsilon)\,ds\,d\tau\qquad(0\leqslant r\leqslant\delta) \tag{2.15}$$

It is easy to prove from the definition of $\psi_1$ and Fubini-Tonelli Theorem that $\psi_1$ satisfies $\varepsilon\psi_{1rr} - l_2\psi_{1r} + m_1\chi_1=0$, and $\psi_1$, $\psi_{1r}>0$ and $\psi_1=O(\varepsilon)$, $\psi_{1r}=O(1)>0$ . Hence, for $\gamma$ large enough and $\varepsilon$ sufficiently small we have (2.14) which is larger than or equal to zero.

In $J^{\delta/2}\cap\Omega_1$ , from the condition (2.5) and the Mean Value Theorem we have

$$
\begin{aligned}
&\varepsilon\Delta\alpha_1-f_1(x,\alpha_1,y_2,\nabla\alpha_1,\varepsilon)\\
&\quad\geqslant-\varepsilon\zeta_{1rr}-\varepsilon\Lambda\zeta_{1r}-\varepsilon\psi_{2rr}+k_1(\zeta_1+\psi_2+\varepsilon\gamma\exp[\lambda h(x)])\\
&\qquad-m_1(\chi_2+\varepsilon\gamma\exp[\lambda h(x)])+l_1(\zeta_{1r}+\psi_{2r})\\
&\qquad+\varepsilon\lambda\gamma\exp[\lambda h(x)]+O(\varepsilon)+O(\varepsilon^2)\\
&\quad\geqslant\kappa_1\zeta_{1r}-\varepsilon\Lambda\zeta_{1r}-(\varepsilon\psi_{2rr}-l_1\psi_{2r}+m_1\chi_2)\\
&\qquad+\varepsilon\gamma k_1\exp[\lambda h(x)]+O(\varepsilon)+O(\varepsilon^2)
\end{aligned}
\tag{2.16}
$$

where we define

$$\psi_2(r,\varepsilon)=\varepsilon\int_{-\infty}^{r/\varepsilon}\exp[l_1\tau]\int_{\tau}^{0}\exp[-l_1s]m_1\chi_2(s,\varepsilon)\,ds\,d\tau \quad (-\delta\leqslant r\leqslant 0)$$

and note that $\psi_2$ satisfies the equation $\varepsilon\psi_{2rr}-l_1\psi_{2r}+m_1\chi_2=0$ . Also, one can use the Fubini-Tonelli Theorem to show that since $\chi_2\in L_1(0,\infty)$, $\psi_{2r}\in L_1[0,\infty)$ and $\psi_2=O(\varepsilon)$ . Also, note that $\psi_2$ and $\psi_{2r}$ are nonnegative. Since the positive term $\kappa_1\zeta_{1r}$ is dominant in (2.16) in the interior layer and $\varepsilon\gamma k_1\exp[\lambda h(x)]$ is dominant in (2.16) in the outer layer, it is enough to show that (2.16) is nonnegative by choosing a sufficiently large $\gamma$ and small enough $\varepsilon$.

Now, consider the intermediate interval $J^{3\delta/4}-J^{\delta/2}$ . Since $\bar{\xi}_1$ and $\bar{\chi}_1$ are transcendentally small terms as $\varepsilon\to 0$ , similar to the previous calculations, we can prove that $\alpha_1$ satisfies the requisited inequalities. The details are omitted.

The next step is to show that $\beta_1$ is an upper solution for $y_1$ in $\Omega_1\cup\Omega_2$. Using a calculation much like the previous proof for $\alpha_1$, by choosing a large value of $\gamma$ and a sufficiently small value of $\varepsilon$, we can show that $\varepsilon\Delta\beta_1-f_1(x,\beta_1,y_2,\nabla\beta_1,\varepsilon)$ is less than or equal to zero in $\Omega_1\cup\Omega_2$.

Finally, we construct upper and lower solutions for $y_2$ in $\Omega$, subject to

$$
\begin{aligned}
&u_2(x)-\varepsilon\gamma\exp[\lambda h(x)]-O(\varepsilon)\leqslant y_1\\
&\quad\leqslant u_2(x)+\bar{\xi}_2(x)\varepsilon)+\varepsilon\gamma\exp[\lambda h(x)]+O(\varepsilon) \qquad (x\in\Omega_2)
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
&u_1(x)-\xi_1(x,\varepsilon)-\varepsilon\gamma\exp[\lambda h(x)]-O(\varepsilon)\leqslant y_2\\
&\quad\leqslant u_1(x)+\varepsilon\gamma\exp[\lambda h(x)]+O(\varepsilon) \qquad (x\in\Omega_1)
\end{aligned}
\tag{2.18}
$$

Let

$$\alpha_2(x,\varepsilon)=\begin{cases} v_2(x)-\xi_1(x,\varepsilon)-\bar{\psi}_3(x,\varepsilon)-\varepsilon\gamma\exp[\lambda h(x)] & (x\in\Omega_2)\\ v_1(x)-\bar{\psi}_4(x,\varepsilon)-\varepsilon\gamma\exp[\lambda h(x)] & (x\in\bar{\Omega}_1)\end{cases}$$

$$\beta_2(x,\varepsilon)=\begin{cases} v_2(x)+\bar{\psi}_3(x,\varepsilon)+\varepsilon\gamma\exp[\lambda h(x)] & (x\in\Omega_2)\\ v_1(x)+\bar{\chi}_2(x,\varepsilon)+\bar{\psi}_4(x,\varepsilon)+\varepsilon\gamma\exp[\lambda h(x)], & x\in\bar{\Omega}_1\end{cases}$$

where the $O(\varepsilon)$ term is independent of $\gamma$, and $\bar{\psi}_3$ and $\bar{\psi}_4$ are smooth positive $O(\varepsilon)$ functions of the form:

$$\begin{cases} \bar{\psi}_3(x,\varepsilon)=\varpi\psi_3 & (x\in J^\delta\cap\Omega_2), \quad \bar{\psi}_3=0 \quad (x\in\Omega_2-J^\delta) \\ \psi_3(r,\varepsilon)=\varepsilon\int_0^{r/\varepsilon}\exp[l_2\tau]\int_\tau^\infty\exp[-l_2 s]m_2\zeta_2(s,\varepsilon)ds d\tau & (0\leqslant r\leqslant\delta) \end{cases} \tag{2.19}$$

$$\begin{cases} \bar{\psi}_4(x,\varepsilon)=\varpi\psi_4 & (x\in J^\delta\cap\Omega_2), \quad \bar{\psi}_4=0 \quad (x\in\Omega_1-J^\delta) \\ \psi_4(r,\varepsilon)=\varepsilon\int_{-\infty}^{r/\varepsilon}\exp[l_1\tau]\int_\tau^0\exp[-l_1 s]m_1\zeta_1(s,\varepsilon)ds d\tau & (-\delta\leqslant r\leqslant 0) \end{cases} \tag{2.20}$$

The verification that $\alpha_2$ and $\beta_2$ satisfy the requisited relationships in Theorem 1 for $\varepsilon$ sufficiently small is similar to the previous proof for $\alpha_1$, $\beta_1$. The details will be omitted.

Put everything together, we can conclude that there exists a solution $y=y(x,\varepsilon)$ of the problem (1.1), (1.2) such that (2.2) and (2.3) are correct. This completes the proof of Theorem 2.

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