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Existence results for unilateral contact problem with friction of thermo-electro-elasticity*

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Abstract This work studies a mathematical model describing the static process of contact between a piezoelectric body and a thermally-electrically conductive foundation. The behavior of the material is modeled with a thermo-electro-elastic constitutive law. The contact is described by Signorini's conditions and Tresca's friction law including the electrical and thermal conductivity conditions. A variational formulation of the model in the form of a coupled system for displacements, electric potential, and temperature is derived. Existence and uniqueness of the solution are proved using the results of variational inequalities and a fixed point theorem.

Key words static frictional contact, thermo-piezoelectric material, Signorini's condition, Tresca's friction, frictional heat generation, variational inequality, fixed point process

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1 Introduction

In 1880, the Curie brothers discovered the piezoelectric effect, that is, the ability of some crystalline material to produce an electrical voltage proportional to the mechanical stress which deforms it. The deformation resulting from the application of an electric potential is the reversible effect. It was suggested by Lippman in 1881 and confirmed experimentally by the Curie. The two effects are the basis for an extensive use of piezoelectric materials in many engineering applications such as sensors, actuators, and intelligent structures.

The electro-elastic characteristics of piezoelectric materials have been studied extensively, and their dependence on temperature is well established[1–5]. Currently, it is interesting to incorporate the thermal effects in addition to the piezoelectric effects. This thermo-piezoelectric model was first proposed by Mindlin[6] and Migórski[7], and the physical laws for the thermo-piezoelectric materials were investigated by Nowacki[8–9]. Chandrasekharaiah[10] generalized Mindlin's theory of thermo-piezoelectricity to some special model, and Tiersten[11] developed the general nonlinear theory of thermo-piezoelasticity.

In the literature, there are few mathematical results dealing with contact problems involving coupling between mechanical and electrical properties[1,2,4,6,7,12–14] and the references therein.

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Up to date, no work has dealt with coupling between thermal effects and piezoelectrical properties. Therefore, there is a need to extend mathematical analysis to this kind of model problems.

This work deals with a new mathematical model which describes the frictional contact between a thermo-piezoelectric body and a conductive foundation. The novelty of this model lies in the chosen thermo-electro-elastic behavior for the body and in the electrical and thermal conditions describing the contact. The motivation of this approach is that the thermal effects, such as thermal deformation and pyroelectric effects, are especially important for many smart ceramic materials. Thus, it may be impossible to predict the electromechanical behavior without taking account of these thermal effects.

Here, we study a static problem of frictional contact under small deformation hypothesis, wherein the behavior of the material is modeled by a nonlinear thermo-electro-elastic constitutive law, and the contact is described by Signorini's condition, Tresca's friction law, and a regularized electrical and thermal conductivity condition. The variational formulation of this problem is derived, and its unique weak solvability is established.

The paper is structured as follows. In Section 2, we state the model of equilibrium process of the thermo-elctro-elastic body in frictional contact with a conductive rigid foundation. In Section 3, we introduce the notation and the assumptions on the problem data. We also derive the variational formulation of the problem, and the main result is stated in Theorem 3.1. In Section 4, we prove the existence of a weak solution to the model and its uniqueness under additional assumptions. The proof is based on an abstract result on elliptic variational inequalities and fixed point arguments.

2 Mathematical model

We consider a piezoelectric body that occupies an open bounded subset Ω in $\mathbb{R}^d\,(d=2,3)$ with a sufficiently smooth boundary $\Gamma=\partial\Omega$. This boundary is divided into three open disjoint parts Γ_D, Γ_N, and Γ_C on one hand and a partition of $\Gamma_D\cup\Gamma_N$ into two open parts Γ_a and Γ_b on the other hand, such that the two parts Γ_D and Γ_a have a nonnegative measure. The body is subjected to the action of body forces of density f_0, a volume electric charges of density q_0, and a heat source of constant strength q_{t}. It is also subjected to mechanical, electrical, and thermal constraints on the boundary. Indeed, the body is assumed to be clamped in Γ_D, and therefore the displacement field vanishes there. Moreover, we assume that a density of traction forces, denoted by f_N, acts on the boundary part Γ_N. We also assume that the electrical potential vanishes on Γ_a, and a surface electrical charge of density q_b is prescribed on Γ_b. Finally, we assume that the temperature θ_0 is prescribed on the surface $\Gamma_N\cup\Gamma_D$.

In the reference configuration, the body may come in contact over Γ_C with an electrically-thermally conductive foundation. We assume that its potential and temperature are maintained at φ_{F} and θ_{F}. The contact is frictional, and there may be electrical charges and heat transfer on the contact surface. The normalized gap between Γ_C and the rigid foundation is denoted by g.

Here and below, we do not indicate the dependence of various functions on the spatial variable $x\in\overline{\Omega}$, the summation convention over repeated indices is used, and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable.

We use S^d for the linear space of second-order symmetric tensors on $\mathbb{R}^d$ and use " $\cdot$ " and $\|\cdot\|$ to represent the inner products and the Euclidean norms on $\mathbb{R}^d$ and S^d,

$$u\cdot v \;=\; u_i\cdot v_i,\quad \|v\|=(v\cdot v)^{\frac{1}{2}},\quad \forall u,\ v\in\mathbb{R}^d,$$

$$\sigma\cdot\tau=\sigma_{ij}\cdot\tau_{ij},\quad \|\tau\|=(\tau\cdot\tau)^{\frac{1}{2}},\quad \forall\sigma,\ \tau\in S^d.$$

We denote by $u : \Omega \to \mathbb{R}^d$ the displacement field, $\sigma : \Omega \to S^d$ and $\sigma = (\sigma_{ij})$ the stress tensor, $\theta : \Omega \to \mathbb{R}$ the temperature, $q : \Omega \to \mathbb{R}^d$ and $q = (q_i)$ the heat flux vector, and by $D : \Omega \to \mathbb{R}^d$ and $D = (D_i)$ the electric displacement field. We also denote $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is the electric potential. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and "Div" and "div" denote the divergence operators for tensor and vector valued functions, respectively, i.e., $\operatorname{Div}\sigma = (\sigma_{ij,j})$ and $\operatorname{div}\xi = (\xi_{j,j})$. We shall adopt the usual notations for normal and tangential components of displacement vector and stress: $v_n = v \cdot n$, $v_\tau = v - v_n n$, $\sigma_n = (\sigma n) \cdot n$, and $\sigma_\tau = \sigma n - \sigma_n n$, where n denotes the outward normal vector on Γ.

We suppose that the process is static. The equations of stress equilibrium, the equation of quasi-stationary electric field, and the heat conduction equation are, respectively, given by

$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{div} D = q_0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\operatorname{div} q = q_{\mathrm{t}} \quad \text{in} \quad \Omega. \tag{3}$$

For the linear piezoelectric material including the thermal expansion effect, we have the following constitutive relations:

$$\sigma = \mathfrak{F}\,\varepsilon(u) - \mathcal{E}^* E(\varphi) - \theta\mathcal{M} \quad \text{in} \quad \Omega, \tag{4}$$

$$D = \mathcal{E}\,\varepsilon(u) + \beta\, E(\varphi) - \theta\mathcal{P} \quad \text{in} \quad \Omega, \tag{5}$$

where $\mathfrak{F} = (f_{ijkl})$, $\mathcal{E} = (e_{ijk})$, $\mathcal{M} = (m_{ij})$, $\beta = (\beta_{ij})$, and $\mathcal{P} = (p_i)$ are, respectively, elastic, piezoelectric, thermal expansion, electric permittivity, and pyroelectric tensors, and $\mathcal{E}^*$ is the transpose of $\mathcal{E}$ given by

$$\mathcal{E}^* = (e^*_{ijk}),$$

where

$$e^*_{ijk} = e_{kij},$$

and

$$\mathcal{E}\sigma v = \sigma\mathcal{E}^* v, \quad \forall\sigma \in S^d,\ v \in \mathbb{R}^d. \tag{6}$$

The Fourier law of heat conduction is given by

$$q = -\mathcal{K}\nabla\theta, \tag{7}$$

where $\mathcal{K} = (k_{ij})$ denotes the thermal conductivity tensor.

Next, to complete the mathematical model according to the description of the physical setting, we have the following boundary conditions:

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{8}$$

$$\sigma\nu = f_N \quad \text{on} \quad \Gamma_N, \tag{9}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{10}$$

$$D \cdot \nu = q_2 \quad \text{on} \quad \Gamma_b, \tag{11}$$

$$\theta = 0 \quad \text{on} \quad \Gamma_N \cup \Gamma_D. \tag{12}$$

We model the frictional contact on Γ_C with Signorini's conditions and Tresca's law

$$\sigma_\nu(u) \leqslant 0, \quad u_\nu - g \leqslant 0, \quad \sigma_\nu(u)\,(u_\nu - g) = 0 \quad \text{on} \quad \Gamma_C, \tag{13}$$

$$\left.\begin{array}{l} \|\sigma_\tau\| \leqslant S \\ \|\sigma_\tau\| < S \Rightarrow u_\tau = 0 \\ \|\sigma_\tau\| = S \Rightarrow \exists \lambda \neq 0,\, \sigma_\tau = -\lambda u_\tau \end{array}\right\} \quad \text{on} \quad \Gamma_C, \tag{14}$$

and with the following regularized electrical and thermal conditions[13,15]:

$$D \cdot \nu = \psi(u_\nu - g)\phi_L(\varphi - \varphi_{\mathrm{F}}) \quad \text{on} \quad \Gamma_C, \tag{15}$$

$$q \cdot \nu = k_{\mathrm{c}}(u_\nu - g)\phi_L(\theta - \theta_{\mathrm{F}}) \quad \text{on} \quad \Gamma_C \tag{16}$$

such that

$$\phi_L(s) = \begin{cases} -L & \text{if} \quad s < -L, \\ s & \text{if} \quad |s| \leqslant L, \\ L & \text{if} \quad s > L, \end{cases}$$

$$\psi(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ k_{\mathrm{e}}\delta r & \text{if} \quad 0 \leqslant r \leqslant 1/\delta, \\ k_{\mathrm{e}} & \text{if} \quad r > 1/\delta, \end{cases}$$

where L is a large positive constant, $\delta > 0$ is a small parameter, and $k_{\mathrm{e}} \geqslant 0$ is the electrical conductivity coefficient such that the thermal conductance function $k_{\mathrm{c}} : r \to k_{\mathrm{c}}(r)$ is supposed to be zero for $r < 0$ and positive otherwise, nondecreasing and Lipschitz continuous. We note that when $\psi \equiv 0$, the equality (15) leads to the condition

$$D \cdot \nu = 0 \quad \text{on} \quad \Gamma_C,$$

which models the case when the foundation is a perfect electric insulator.

Similarly, we describe the case of a perfect thermally insulating foundation with

$$q \cdot \nu = 0 \quad \text{on} \quad \Gamma_C.$$

We collect the above equations and conditions to obtain the following mathematical problem.

Problem (P) Find a displacement field $u : \Omega \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \to \mathbb{R}$, and a temperature field $\theta : \Omega \to \mathbb{R}$ such that (1)–(16) hold.

Note that if the displacement field u, the electric potential φ, and the temperature θ which solve Problem (P) are known, then the stress tensor σ, the electric displacement field D, and the heat flux q can be obtained from (4)–(7).

3 Variational formulation and main result

In this section, we derive a weak formulation of Problem (P) and investigate its solvability. Everywhere in what follows we use the standard notation for the L^p spaces associated with Ω

and Γ. We also use the Hilbert spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d,$$

$$\mathcal{H} = \{\sigma = \sigma_{ij},\, \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}$$

endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \mathrm{d}x, \quad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ji} \mathrm{d}x,$$

$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}.$$

Keeping in mind the boundary condition (8), we introduce the closed subspace of H_1

$$V = \{v \in H_1 \,|\, v = 0 \text{ on } \Gamma_D\}$$

and the set of admissible displacements

$$K = \{v \in V \,|\, v_\nu - g \leqslant 0 \text{ on } \Gamma_C\}.$$

Here and below, we write w for the trace γw of the function $w \in H_1$ on Γ. Since $\mathrm{meas}(\Gamma_D) > 0$, Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \geqslant c_k \|v\|_{H_1}, \quad \forall v \in V, \tag{17}$$

where c_k is a nonnegative constant depending only on Ω and Γ_D. Therefore, the space V endowed with the inner product $(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$ is a real Hilbert space, and its associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$ is equivalent on V to the usual norm $\|\cdot\|_{H_1}$. By Sobolev's trace theorem, there exists a constant $c_0 > 0$ which depends only on Ω, Γ_C, and Γ_D such that

$$\|v\|_{L^2(\Gamma)^d} \leqslant c_0 \|v\|_V, \quad \forall v \in V. \tag{18}$$

We also introduce the function spaces

$$W = \{\psi \in H^1(\Omega) |\, \psi = 0 \text{ on } \Gamma_a\}, \quad Q = \{\, \theta \in H^1(\Omega) |\, \theta = 0 \text{ on } \Gamma_D \cup \Gamma_N\}.$$

Similarly, we write ζ for the trace $\gamma\zeta$ of the function $\zeta \in H^1(\Omega)$ on Γ. Since $\mathrm{meas}(\Gamma_a) > 0$ and $\mathrm{meas}(\Gamma_D) > 0$, it is known that W and Q are real Hilbert spaces with the inner products. $(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_H$, and $(\theta, \xi)_Q = (\nabla\theta, \nabla\xi)_H$. Moreover, the associated norms $\|\psi\|_W = \|\nabla\psi\|_H$ and $\|\xi\| = \|\nabla\xi\|_H$ are equivalent on W and Q, respectively, with the usual norms $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem, there exists a constant $c_1 > 0$ which depends only on Ω, Γ_a, and Γ_C such that

$$\|\psi\|_{L^2(\Gamma_C)} \leqslant c_1 \|\psi\|_W, \quad \forall \psi \in W, \tag{19}$$

and a constant $c_2 > 0$ which depends only on Ω, Γ_D, Γ_N, and Γ_C such that

$$\|\xi\|_{L^2(\Gamma_C)} \leqslant c_2 \|\xi\|_Q, \quad \forall \xi \in Q. \tag{20}$$

In the study of the mechanical problem (P), we need the following assumptions.

(h$_1$) The elasticity operator $\mathfrak{F} : \Omega \times S^d \to S^d$, the electric permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, and the thermal conductivity tensor $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the usual properties of symmetry, boundedness, and ellipticity,

$$f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega), \quad \beta_{ij} = \beta_{ji} \in L^\infty(\Omega), \quad k_{ij} = k_{ji} \in L^\infty(\Omega),$$

and there exists that $m_{\mathfrak{F}}, m_\beta, m_{\mathcal{K}} > 0$ such that

$$f_{ijkl}(x)\ \xi_k\xi_l \geqslant m_{\mathfrak{F}}\ \|\xi\|^2, \quad \forall \xi \in S^d, \quad \forall x \in \Omega,$$
$$\beta_{ij}\ \zeta_i\zeta_j \geqslant m_\beta\ \|\zeta\|^2, \quad k_{ij}\ \zeta_i\zeta_j \geqslant m_{\mathcal{K}}\ \|\zeta\|^2 \quad \text{for all } \zeta \in \mathbb{R}^d.$$

(h_2) The piezoelectric tensor $\mathcal{E} = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d$, the thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, and the pyroelectric tensor $\mathcal{P} = (p_i) : \Omega \to \mathbb{R}^d$ satisfy

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega), \quad m_{ij} = m_{ji} \in L^\infty(\Omega), \quad p_i \in L^\infty(\Omega).$$

(h_3) The surface electrical conductivity $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$ and the thermal conductance $k_c : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$ satisfy the following hypothesis for $\pi = \psi, k_c$: $\exists\, M_\pi > 0$ such that $|\pi(x,u)| \leqslant M_\pi$, $\forall u \in \mathbb{R}$, a.e. $x \in \Gamma_C$, and $x \to \pi(x,u)$ is mesurable on Γ_C for all $u \in \mathbb{R}$ and is zero for all $u \leqslant 0$.

(h_4) The function $u \to \pi(x,u)$ ($\pi = \psi, k_c$) is a Lipschitz function on $\mathbb{R}$ for all $x \in \Gamma_C$. $|\pi(x,u_1) - \pi(x,u_2)| \leqslant L_\pi |u_1 - u_2|$, $\forall u_1, u_2 \in \mathbb{R}$, where L_π is a positive constant.

(h_5) The forces, the traction, the volume, the surface charge densities, and the strength of the heat source satisfy

$$f_0 \in L^2(\Omega)^d, \quad f_N \in L^2(\Gamma_N)^d, \quad q_0 \in L^2(\Omega), \quad q_b \in L^2(\Gamma_b), \quad q_t \in L^2(\Omega).$$

(h_6) The potential and temperature of the contact surface satisfy

$$\varphi_F \in L^2(\Gamma_C), \quad \theta_F \in L^2(\Gamma_C).$$

(h_7) The friction bound function satisfies

$$S \in L^\infty(\Gamma_C), \quad S \geqslant 0.$$

Next, using Riesz's representation theorem, we define the elements $f \in V$, $q \in W$, and $q_{th} \in Q$ by

$$(f, v)_V = \int_\Omega f_0 \cdot v \mathrm{d}x + \int_{\Gamma_N} f_N \cdot v \mathrm{d}a, \quad \forall v \in V, \tag{21}$$

$$(q_e, \xi)_W = \int_\Omega q_0\, \xi \mathrm{d}x - \int_{\Gamma_b} q_b\, \xi \mathrm{d}a, \quad \forall \xi \in W, \tag{22}$$

$$(q_{th}, \eta)_Q = \int_\Omega q_t\, \eta \mathrm{d}x, \quad \forall \eta \in W, \tag{23}$$

and we define the mappings $j : V \to \mathbb{R}$, $\ell : V \times W^2 \to \mathbb{R}$, and $\chi : V \times Q^2 \to \mathbb{R}$ by

$$j(v) = \int_{\Gamma_C} S\ \|v_\tau\| \mathrm{d}a, \quad \forall v \in V, \tag{24}$$

$$\ell(u, \varphi, \xi) = \int_{\Gamma_C} \psi(u_\nu - g)\phi_L(\varphi - \varphi_F)\xi \mathrm{d}a, \quad \forall u \in V, \quad \forall \varphi,\ \xi \in W, \tag{25}$$

$$\chi(u, \theta, \eta) = \int_{\Gamma_C} k_c(u_\nu - g)\phi_L(\theta - \theta_F)\eta \mathrm{d}a, \quad \forall u \in V, \quad \forall \theta,\ \eta \in Q, \tag{26}$$

respectively.

It follows from the assumptions (h_3) and (h_6)–(h_7) that the integrals below are well-defined.

Now, by a standard variational technique, it is straightforward to see that if $(u,\ \varphi,\ \theta)$ satisfies the conditions (1)–(16), then

$$\big(\sigma, \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + j(v) - j(u) \geqslant (f, v-u)_V, \quad \forall v \in K, \tag{27}$$

$$\big(D, \nabla\xi\big)_H = \ell(u, \varphi, \xi) - (q_{\mathrm{e}}, \xi)_W, \quad \forall \xi \in W, \tag{28}$$

$$\big(q, \nabla\eta\big)_H = \chi(u, \varphi, \eta) - (q_{\mathrm{th}}, \eta)_Q, \quad \forall \eta \in Q. \tag{29}$$

We plug (4) in (27), (5) in (28), and (7) in (29) and use $E = -\nabla\varphi$ to obtain the variational formulation of P in terms of displacement, electric potential, and temperature.

Problem (PV) Find a displacement field $u \in K$, an electric potential $\varphi \in W$, and a temperature field $\theta : \Omega \to \mathbb{R}$ such that

$$\begin{aligned}&\big(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + \big(\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u)\big)_H - \big(\mathcal{M}\theta, \varepsilon(v) - \varepsilon(u)\big)_H \\ &+ j(v) - j(u) \geqslant\ (f, v-u)_V, \quad \forall v \in K,\end{aligned} \tag{30}$$

$$\big(\beta\nabla\varphi, \nabla\xi\big)_H - \big(\mathcal{E}\varepsilon(u), \nabla\xi\big)_H - \big(\mathcal{P}\nabla\theta, \nabla\xi\big)_H + \ell(u, \varphi, \xi) = (q_{\mathrm{e}}, \xi)_W, \quad \forall \xi \in W, \tag{31}$$

$$\big(\mathcal{K}\nabla\theta, \nabla\eta\big)_H + \chi(u, \theta, \eta) = (q_{\mathrm{th}}, \eta)_Q, \quad \forall \eta \in Q. \tag{32}$$

Now, we are able to state the following main result of existence and uniqueness.

Theorem 3.1 *Assume that the assumptions* (h_1)–(h_3) *and* (h_5)–(h_7) *hold. Then,* (1) *Problem* (PV) *has at least one solution* $(u, \varphi, \theta) \in K \times W \times Q$; (2) *under the assumption* (h_4), *there exists* $L^* > 0$ *such that if*

$$M_\psi + M_{k_{\mathrm{c}}} + LL_\psi + LL_{k_{\mathrm{c}}} + \max(m^*, p^*) < L^*,$$

then Problem (PV) *has a unique solution with* $m^* = \sup\limits_{ij} \|m_{ij}\|$ *and* $p^* = \sup\limits_i \|p_i\|$.

The proof of our main result will be presented in the next section.

4 Proof of main result

The proof of Theorem 3.1 will be carried out in several steps, and it is based on arguments of variational inequalities and Schauder's fixed point theorem. To this end, we assume in the following that (h_1)–(h_3) and (h_5)–(h_7) hold.

Let $\mathcal{K}_1$ and $\mathcal{K}_1$ denote two closed convex sets of $L^2(\Gamma_C)$ as follows:

$$\mathcal{K}_1 = \{z_1\ \in L^2(\Gamma_C), \quad \|z_1\|_{L^2(\Gamma_C)} \leqslant k_1\},$$

$$\mathcal{K}_2 = \{z_2\ \in L^2(\Gamma_C), \quad \|z_2\|_{L^2(\Gamma_C)} \leqslant k_2\}$$

with k_1 and k_2 to be specified later, and let $z = (z_1, z_2) \in L^2(\Gamma_C)^2$ be given. We define the functions

$$\ell_1(z, \xi) = \int_{\Gamma_C} z_1 \xi \mathrm{d}a, \quad \forall \xi \in W, \tag{33}$$

$$\chi_2(z, \eta) = \int_{\Gamma_C} z_2 \eta \mathrm{d}a, \quad \forall \eta \in Q. \tag{34}$$

In the first step, we consider the following variational problem.

Problem (PV$_z$) Find a displacement field $u_z \in K$, an electric potential $\varphi_z \in W$, and a temperature field $\theta_z \in Q$ such that

$$(\mathfrak{F}\varepsilon(u_z), \varepsilon(v) - \varepsilon(u_z))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_z, \varepsilon(v) - \varepsilon(u_z))_H - (\mathcal{M}\theta_z, \varepsilon(v) - \varepsilon(u_z))_H$$

$$+ j(v) - j(u_z) \geqslant (f, v - u_z)_V, \quad \forall v \in K, \tag{35}$$

$$(\beta\nabla\varphi_z, \nabla\xi)_H - (\mathcal{E}\varepsilon(u_z), \nabla\xi)_H - (\mathcal{P}\theta_z, \nabla\xi)_H = (q_{\mathrm{e}}, \xi)_W - \ell_1(z, \xi), \quad \forall \xi \in W, \tag{36}$$

$$(\mathcal{K}\nabla\theta_z, \nabla\eta)_H = (q_{\mathrm{th}}, \eta)_Q - \chi_2(z, \eta), \quad \forall \eta \in Q. \tag{37}$$

The coupling system leads to considerable difficulties in the analysis of the problem. Therefore, we first solve the following thermic problem with the unknown θ_z:

$$(\mathrm{PV}_z^\theta): \quad (\mathcal{K}\nabla\theta_z, \nabla\eta)_H = (q_{\mathrm{th}}, \eta)_Q - \chi_2(z, \eta), \quad \forall \eta \in Q.$$

To this end, using Riesz's representation theorem, we find the element $q_z \in Q$ and the operator $\mathcal{T} : Q \to Q$ defined by

$$\begin{cases} (q_z, \eta)_Q = (q_{\mathrm{th}}, \eta)_Q - \chi_2(z, \eta), \quad \forall \eta \in Q, \\ (\mathcal{T}\theta_z, \eta)_Q = (\mathcal{K}\nabla\theta_z, \nabla\eta)_H, \quad \forall \eta \in Q. \end{cases} \tag{38}$$

Thus, Problem (PV$_z^\theta$) can be also written in the following form:

$$\text{Find } \theta_z \in Q \quad \text{such that} \quad (\mathcal{T}\theta_z, \eta)_Q = (q_z, \eta)_Q, \quad \forall \eta \in Q. \tag{39}$$

From (38), (34), (23), and (h$_5$), it is easy to see that the linear form $\eta \to (q_z, \eta)_Q$ is continuous on Q. We note that, under the assumption (h$_1$) on $\mathcal{K}$, the operator $\mathcal{T}$ is linear symmetric and positive definite on Q. Moreover, $\mathcal{T}$ is a linear continuous invertible operator on Q, and let $\mathcal{C} = \mathcal{T}^{-1}$. Thus, by the Lax-Milgram theorem, we conclude that (39) has a unique solution

$$\theta_z = \mathcal{C}q_z \in Q \quad \text{satisfying} \quad \|\theta_z\|_Q \leqslant \frac{1}{m_{\mathcal{K}}}\, \|q_z\|_Q. \tag{40}$$

Now, we substitute $\theta_z = \mathcal{C}q_z$ in (35) and (36) and get

$$(\mathfrak{F}\varepsilon(u_z), \varepsilon(v) - \varepsilon(u_z))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_z, \varepsilon(v) - \varepsilon(u_z))_H + j(v) - j(u_z)$$

$$\geqslant (f_z, v - u_z)_V, \quad \forall v \in K, \tag{41}$$

$$(\beta\nabla\varphi_z, \nabla\xi)_H - (\mathcal{E}\varepsilon(u_z), \nabla\xi)_H = (q_z, \xi)_W, \quad \forall \xi \in W, \tag{42}$$

where

$$(f_z, v)_V = (f, v)_V + (\mathcal{M}\mathcal{C}q_z, \varepsilon(v))_H, \quad \forall v \in K, \tag{43}$$

$$(q_z, \xi)_W = (q_{\mathrm{e}}, \xi)_W + (\mathcal{P}\mathcal{C}q_z, \nabla\xi)_H - \ell_1(z, \xi), \quad \forall \xi \in W. \tag{44}$$

In order to solve (41)–(42), we consider the product spaces $X = V \times W$ and $Y = (L^2(\Gamma_C))^2$ endowed with the inner products

$$(x, y)_X = (u, v)_V + (\varphi, \xi)_W, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{45}$$

$$(z, \zeta)_Y = (z_1, \zeta_1)_{L^2(\Gamma_C)} + (z_2, \zeta_2)_{L^2(\Gamma_C)}, \quad \forall z = (z_1, z_2),\ \zeta = (\zeta_1, \zeta_2) \in Y, \tag{46}$$

and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $U = K \times W$ be a non-empty closed convex subset of X. We also introduce the operator $A : X \to X$ and the function $J : U \to \mathbb{R}$ defined by

$$(Ax, y)_X = (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\beta\nabla\varphi, \nabla\xi)_H + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_H$$

$$- (\mathcal{E}\varepsilon(u), \nabla\xi)_H, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{47}$$

$$J(y) = j(v) = \int_{\Gamma_C} S\|v_\tau\| \mathrm{d}a, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{48}$$

$$f_z^{\mathrm{e}} = (f_z, q_{\mathrm{e}z}) \in X. \tag{49}$$

Keeping in mind the proprieties of the operators $\mathcal{T}$, $\mathcal{M}$, and $\mathcal{P}$, it follows that $\mathcal{MC}$ and $\mathcal{PC}$ are linear continuous operators. Hence, by Riesz's representation theorem and (49), we deduce that $f_z \in V$ and $q_z \in W$, then we conclude that $f_z^{\mathrm{e}} \in X$.

We start by the following equivalence result.

Lemma 4.1 *The pair* $x_z = (u_z, \varphi_z) \in U$ *is a solution to* (41)–(42) *if and only if*

$$(Ax_z, y - x_z)_X - J(y) - J(x_z) \geqslant (f_z^{\mathrm{e}}, y - x_z)_X, \quad \forall y = (v, \xi) \in U. \tag{50}$$

Proof We use $(\xi - \varphi_z)$ in (42) and add the corresponding inequality to (41). Then, we deduce (50). Conversely, let $x_z = (u_z, \varphi_z) \in U$ be a solution of the elliptic variational inequality (50). By taking $y = (v, \varphi_z)$ in (50), where v is an arbitrary element of K, we obtain (41). Moreover, if we take successively $y = (v, \varphi_z + \xi)$ and $y = (v, \varphi_z - \xi)$ in (50), where ξ is an arbitrary element of W, we will obtain (42), which finishes the proof.

We now use Lemma 4.1 to obtain the following existence and uniqueness result.

Lemma 4.2 *For any* $z \in \mathcal{K}_1 \times \mathcal{K}_2$ *assumed to be known, we have*

(i) *under the assumptions* (h_1)–(h_2) *and* (h_5)–(h_7), (50) *has a unique solution* $x_z = (u_z, \varphi_z) \in K \times W$, *and there exists* $c > 0$ *such that* $\|x_z\|_X \leqslant c\, \|f_z^{\mathrm{e}}\|_X$;

(ii) *the solution* x_z *of* (50) *depends on* $z \in Y$ *Lipschitz continuously.*

Proof We first remark that J is proper and convex on U. Moreover, by (h_7), we find that J is Lipschitz continuous. Therefore, it is a fortiori lower semicontinuous function. Now, we use (h_1), (6), (47), and (45) to see that A is a strongly monotone Lipschitz continuous operator on X, i.e., there exists that $m_A, M_A > 0$ such that $(Ax - Ay, x - y)_X \geqslant m_A\|x - y\|_X^2$ and $\|Ax - Ay\|_X \leqslant M_A\|x - y\|_X$. Since U is a non-empty closed convex set of X, by a standard result on elliptic variational inequalities (see [16]), it follows that there exists a unique element $(u_z, \varphi_z) \in U$ which satisfies (50).

Moreover, if we take $y = 0$ in (50), we get

$$(Ax_z, x_z)_X + J(x_z) \leqslant (f_z^{\mathrm{e}}, x_z)_X.$$

As the friction coefficient $S \geqslant 0$, we have $J(x_z) \geqslant 0$ and thus

$$(Ax_z, x_z)_X \leqslant (f_z^{\mathrm{e}}, x_z)_X.$$

Taking in mind the strong monotony of A, we deduce that there exists $c > 0$ such that

$$\|x_z\|_X \leqslant c\, \|f_z\|_X \quad \text{with} \quad c = \frac{1}{m_A}. \tag{51}$$

For the second part of Lemma 4.2, let us consider two given elements $z=(z_1,z_2)$, $z'=(z'_1,z'_2)$ of Y and the associate solutions x_z , $x_{z'}$ of (50). For all $y\in U$, we have

$$(Ax_z, y-x_z)_X + J(y) - J(x_z) \geqslant (f^{\mathrm{e}}_z, y-x_z)_X,$$

$$(Ax_{z'}, y-x_{z'})_X + J(y) - J(x_{z'}) \geqslant (f^{\mathrm{e}}_{z'}, y-x_{z'})_X.$$

If we take $y=x_{z'}$ in the first inequality and $y=x_z$ in the second one, we obtain

$$(Ax_z, x_{z'}-x_z)_X + J(x_{z'}) - J(x_z) \geqslant (f^{\mathrm{e}}_z, x_{z'}-x_z)_X,$$

$$(Ax_{z'}, x_z-x_{z'})_X + J(x_z) - J(x_{z'}) \geqslant (f^{\mathrm{e}}_{z'}, x_z-x_{z'})_X.$$

Hence,

$$(Ax_z - Ax_{z'}, x_z-x_{z'})_X \leqslant (f^{\mathrm{e}}_z - f^{\mathrm{e}}_{z'}, x_z-x_{z'})_X. \tag{52}$$

Furthermore, we have the following inequality:

$$\begin{aligned}
&(f^{\mathrm{e}}_z - f^{\mathrm{e}}_{z'}, x_z-x_{z'})_X \\
&= \int_{\Gamma_C} (z_1-z'_1)(\varphi_z-\varphi_{z'})\mathrm{d}a + (\mathcal{MC}(q_z-q_{z'}), \varepsilon(u_z)-\varepsilon(u_{z'}))_{\mathcal{H}} \\
&\quad + (\mathcal{PC}(q_z-q_{z'}), \nabla\varphi_z - \nabla\varphi_{z'})_{L^2(\Omega)} \\
&\leqslant \|z_1-z'_1\|_{L^2(\Gamma_C)}\ \|\varphi_z-\varphi_{z'}\|_{L^2(\Gamma_C)} + \frac{m^*}{m_{\mathcal{K}}}\|q_z-q_{z'}\|_{L^2(\Omega)}\|\varepsilon(u_z)-\varepsilon(u_{z'})\|_{\mathcal{H}} \\
&\quad + \frac{p^*}{m_{\mathcal{K}}}\ \|q_z-q_{z'}\|_{L^2(\Omega)}\|\nabla\varphi_z-\nabla\varphi_{z'}\|_{L^2(\Omega)},
\end{aligned}$$

and then

$$\begin{aligned}
&(f^{\mathrm{e}}_z - f^{\mathrm{e}}_{z'}, x_z-x_{z'})_X \\
&\leqslant \|z_1-z'_1\|_{L^2(\Gamma_C)}\|\varphi_z-\varphi_{z'}\|_{L^2(\Gamma_C)} + \frac{c_2\, m^*}{m_{\mathcal{K}}}\ \|z_2-z'_2\|_{L^2(\Omega)}\|\varepsilon(u_z)-\varepsilon(u_{z'})\|_{\mathcal{H}} \\
&\quad + \frac{c_2\, p^*}{m_{\mathcal{K}}}\|z_2-z'_2\|_{L^2(\Omega)}\|\nabla\varphi_z-\nabla\varphi_{z'}\|_{L^2(\Omega)}.
\end{aligned} \tag{53}$$

Combining (52), (53), (19), and (20) yields

$$\begin{aligned}
&(Ax_z - Ax_{z'}, x_z-x_{z'})_X \\
&\leqslant c_1\ \|z_1-z'_1\|_{L^2(\Gamma_C)}\ \|\varphi_z-\varphi_{z'}\|_W + \frac{c_2\, m^*}{m_{\mathcal{K}}}\|z_2-z'_2\|_{L^2(\Gamma_C)}\ \|u_z-u_{z'}\|_V \\
&\quad + \frac{c_2\, p^*}{m_{\mathcal{K}}}\ \|z_2-z'_2\|_{L^2(\Gamma_C)}\|\varphi_z-\varphi_{z'}\|_W.
\end{aligned} \tag{54}$$

The strong monotonicity of A, combined with (54) and (46), implies that there exists a positive constant c_3 depending on the constants $c_1, c_2, m_A, m_{\mathcal{K}}, m^*$, and p^* such that

$$\|x_z - x_{z'}\|_X \leqslant c_3\ \|z-z'\|_Y. \tag{55}$$

Hence, the second part of this lemma is established.

Now, we have the following result.

Lemma 4.3 *For any* $z \in \mathcal{K}_1 \times \mathcal{K}_2$ *assumed to be known and under the assumptions* (h_1)–(h_2) *and* (h_5)–(h_7), *the solution* $\widetilde{x}_z = (u_z, \varphi_z, \theta_z) \in K \times W \times Q$ *of Problem* (PV_z) *depends on* $z \in Y$ *Lipschitz continuously.*

Proof We consider the product space $\mathcal{X} = X \times Q$ endowed with the inner product

$$(\omega, \vartheta)_{\mathcal{X}} = (x, y)_X + (\theta, \eta)_Q, \quad \forall \omega = (x, \theta),\ \vartheta = (y, \eta) \in \mathcal{X} \tag{56}$$

and the associated norm $\|\cdot\|_{\mathcal{X}}$. Now, let $z = (z_1, z_2)$ and $z' = (z_1', z_2')$ be two given elements of Y and θ_z, and let $\theta_{z'}$ be the corresponding solution of (39). Using the linearity of $\mathcal{T}$, we find that

$$\|\theta_{z'} - \theta_z\|_Q \leqslant \frac{1}{m_{\mathcal{K}}} \|q_{z'} - q_z\|_Q.$$

It follows from (20) and (38) that

$$\begin{aligned}(q_{z'} - q_z, \eta)_Q =& \chi_2(z', \eta) - \chi_2(z, \eta)\\ &\leqslant c_2\ \|z_2' - z_2\|_{L^2(\Gamma_C)} \|\eta\|_Q, \quad \forall \eta \in Q.\end{aligned}$$

Then,

$$\|q_{z'} - q_z\|_Q \leqslant\ c_2\ \|z_2' - z_2\|_{L^2(\Gamma_C)}. \tag{57}$$

We combine the previous inequalities to see that

$$\begin{aligned}\|\theta_{z'} - \theta_z\|_Q \leqslant& \frac{c_2}{m_{\mathcal{K}}}\ \|z_2' - z_2\|_{L^2(\Gamma_C)}\\ &\leqslant \frac{c_2}{m_{\mathcal{K}}}\ \|z' - z\|_{L^2(\Gamma_C)^2}.\end{aligned} \tag{58}$$

Lemma 4.3 is now a consequence of (58), (55), and (56).

We now use the properties (h_3) and (h_4) of the constitutive functions ψ and k_{c} to define an operator $\Lambda :\ Y \to L^2(\Gamma_C)^2$ by the formula

$$\Lambda z = \big(\psi(u_\upsilon - g)\phi_L(\varphi - \varphi_{\mathrm{F}}), k_{\mathrm{c}}(u_\upsilon - g)\phi_L(\theta - \theta_{\mathrm{F}})\big). \tag{59}$$

In the second step, we prove that the operator Λ has a fixed point. To this end, we will need the following result.

Lemma 4.4 *The mapping is* $z \to \widetilde{x}_z = (u_z, \varphi_z, \theta_z)$, *where* $\widetilde{x}_z$ *is the solution of Problem* (PV_z), *which is weakly continuous from* Y *to* $V \times W \times Q$.

Proof Let $z_n = (z_{1n}, z_{2n})$ be a sequence of Y which converges weakly to $z = (z_1, z_2)$, and we denote $\widetilde{x}_{z_n} = (x_{z_n}, \theta_{z_n}) \in U \times Q$ with $x_{z_n} = (u_{z_n}, \varphi_{z_n})$, the solution of Problem (PV_z) corresponding to z_n. Using (38), (40), (20), and (46), we have

$$\begin{aligned}\|\theta_{z_n}\|_Q \leqslant& \frac{1}{m_{\mathcal{K}}}\ \|q_{z_n}\|_Q\\ &\leqslant \frac{1}{m_{\mathcal{K}}}(\|q_{\mathrm{th}}\|_Q + c_2\ \|z_{2n}\|_{L^2(\Gamma_C)})\\ &\leqslant \frac{1}{m_{\mathcal{K}}}(\|q_{\mathrm{th}}\|_Q + c_2\ \|z_n\|_Y).\end{aligned}$$

Thus, the sequence (θ_{z_n}) is bounded in Q. Then, there exists $\widetilde{\theta} \in Q$ and a subsequence $(\theta_{z_{n_k}})$ such that $\theta_{z_{n_k}} \rightharpoonup \widetilde{\theta}$. Using (39), we get

$$(\mathcal{T}\widetilde{\theta}, \widetilde{\theta} - \eta) \leqslant \varliminf_{n\to\infty} (\mathcal{T}\theta(z_{n_k}), \theta(z_{n_k}) - \eta) \leqslant (q_z, \widetilde{\theta} - \eta)_Q. \tag{60}$$

Taking $\eta = \widetilde{\theta} \pm \eta^*$ in the previous inequality, we find

$$(\mathcal{T}\widetilde{\theta}, \eta^*) = (q_z, \eta^*)_Q, \quad \forall \eta^* \in Q. \tag{61}$$

According to (39) and (61), we conclude that $\widetilde{\theta}$ is a solution of Problem (PV_z^θ). By the uniqueness of the solution of this variational equality, we deduce that $\widetilde{\theta} = \theta_z$. Since θ_z is the unique limit of any subsequence $(\theta_{z_{n_k}})$, we deduce that the whole sequence (θ_{z_n}) is weakly convergent to θ_z in Q, which ensures the weak continuity of the mapping $z \to \theta_z$. Moreover, x_{z_n} is a solution of (50) implying

$$(Ax_{z_n}, y - x_{z_n})_X + J(y) - J(x_{z_n}) \geqslant (f^{\mathrm{e}}_{z_n}, y - x_{z_n})_X, \quad \forall y \in U. \tag{62}$$

Now, we take $y = 0$ in (62) to obtain

$$(Ax_{z_n}, x_{z_n})_X \leqslant (f^{\mathrm{e}}_{z_n}, x_{z_n})_X - J(x_{z_n}). \tag{63}$$

Keeping in mind (49), (43), and (44), we have

$$\begin{aligned}(f^{\mathrm{e}}_{z_n}, x_{z_n})_X = &(f, u_{z_n})_V + (q_{\mathrm{e}}, \varphi_{z_n})_W + (\mathcal{M}\mathcal{C}\, q_{z_n}, \varepsilon(u_{z_n}))_H \\ &+ (\mathcal{P}\mathcal{C}\, q_{z_n}, \nabla\varphi_{z_n})_H - \ell_1(z_{1_n}, \varphi_{z_n}).\end{aligned}$$

Then,

$$\|f^{\mathrm{e}}_{z_n}\|_X \leqslant \|f\|_V + \|q_{\mathrm{e}}\|_W + \Big(\frac{c_2\, m^*}{m_{\mathcal{K}}} + \frac{c_2\, p^*}{m_{\mathcal{K}}}\Big)\, \|q_{z_n}\|_Q + c_1 \|z_{1n}\|_{L^2(\Gamma_C)}. \tag{64}$$

We also have

$$\|q_{z_n}\|_Q \leqslant \|q_{\mathrm{th}}\|_Q + c_2\ \|z_{2n}\|_{L^2(\Gamma_C)} \tag{65}$$

and

$$J(x_{z_n}) \leqslant c_0\ \|S\|_{L^\infty(\Gamma_C)}\ \|x_{z_n}\|_X. \tag{66}$$

The strong monotonicity of A combined with (64)–(66) implies that there exists a positive constant c_4 which depends on the constants $c_0, c_1, c_2, m_A, m_{\mathcal{K}}, S, m^*$, and p^* of the problem such that

$$\|x_{z_n}\|_X \leqslant c_4\big(\|f\|_V + \|q_{\mathrm{e}}\|_W + \|q_{\mathrm{th}}\|_Q + \|z_n\|_Y\big). \tag{67}$$

Thus, the sequence (x_{z_n}) is bounded in the Hilbert space X. There exists $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi}) \in X$ and a subsequence $(x_{z_{n_k}})$ such that $x_{z_{n_k}} \rightharpoonup \widetilde{x}$. Since $U \subset X$ is a closed convex subset, it is weakly closed and $\widetilde{x} \in U$. Moreover, using the compactness of the trace map $\gamma : X \to L^2(\Gamma_C)^d \times L^2(\Gamma_C)$, it follows from the weak convergence of $(x_{z_{n_k}})$ that

$$(x_{z_{n_k}}) \to \widetilde{x} \text{ strongly in } L^2(\Gamma_C)^d \times L^2(\Gamma_C). \tag{68}$$

Next, let us prove that $\widetilde{x}$ is the solution of (50). We have

$$\begin{aligned}(f^{\mathrm{e}}_{z_n}, y - x_{z_n})_X = &(f, v - u_{z_n})_V + (q_{\mathrm{e}}, \xi - \varphi_{z_n})_W - \ell_1(z_n, \xi - \varphi_{z_n}) \\ &+ (\mathcal{M}\mathcal{C}q_{z_n}, \varepsilon(v) - \varepsilon(u_{z_n}))_H + (\mathcal{P}\mathcal{C}q_{z_n}, \nabla\xi - \nabla\varphi_{z_n})_H.\end{aligned}$$

From (34), (38), and $(z_n) \rightharpoonup z$, we get

$$\begin{cases} (\mathcal{M}\mathcal{C}q_{z_n}, \varepsilon(v) - \varepsilon(u_{z_n}))_H \to (\mathcal{M}\mathcal{C}q_z, \varepsilon(v) - \varepsilon_{z_n})_H, \\ \\ (\mathcal{P}\mathcal{C}q_{z_n}, \nabla\xi - \nabla\varphi_{z_n})_H \to (\mathcal{P}\mathcal{C}q_z, \nabla\xi - \nabla\varphi_z)_H, \end{cases} \tag{69}$$

and from

$$\begin{aligned} |\ell_1(z_n, \xi - \varphi_n) - \ell_1(z_n, \xi - \widetilde{\varphi})| &\leqslant \|z_{1n}\|_{L^2(\Gamma_C)} \ \|\varphi_{z_n} - \widetilde{\varphi}\|_{L^2(\Gamma_C)} \\ &\leqslant \|z_n\|_Y \ \|x_{z_n} - \widetilde{x}\|_{L^2(\Gamma_C)^2}, \end{aligned}$$

we get

$$\ell_1(z_n, \xi - \varphi_{z_n}) \to \ell_1(z, \xi - \widetilde{\varphi}). \tag{70}$$

The two previous results (69) and (70) give

$$(f^{\mathrm{e}}_{z_n}, y - x_{z_n})_X \to (f^{\mathrm{e}}_z, y - \widetilde{x})_X. \tag{71}$$

Furthermore, the inequality

$$\begin{aligned} |\ J(x_{z_n}) - J(\widetilde{x})\ | &\leqslant \|S\|_{L^\infty(\Gamma_C)} \|u_{z_n} - \widetilde{u}\|_{L^2(\Gamma_C)^d} \\ &\leqslant c_0\ \|S\|_{L^\infty(\Gamma_C)} \|x_{z_n} - \widetilde{x}\|_{L^2(\Gamma_C)^2} \end{aligned}$$

gives

$$J(x_{z_n}) \to J(\widetilde{x}). \tag{72}$$

It follows from (62) that

$$(Ax_{z_n}, y - x_{z_n})_X \geqslant (f^{\mathrm{e}}_{z_n}, y - x_{z_n})_X - (J(y) - J(\widetilde{x})) + (J(x_{z_n}) - J(\widetilde{x})).$$

Now, we combine (62), (71), and (72) with the pseudo-monotonicity of A to deduce

$$\begin{cases} \widetilde{x} \in U, \\ (A\widetilde{x}, y - \widetilde{x})_X + J(y) - J(\widetilde{x}) \geqslant (f_z, y - \widetilde{x})_X, \quad \forall y \in U. \end{cases} \tag{73}$$

Thus, we find that $\widetilde{x}$ is a solution of Problem (PV_z), and from the uniqueness of the solution for this variational inequality, we deduce that $\widetilde{x} = x_z$. Moreover, since x_z is the unique weak limit of any subsequence of (x_{z_n}), we obtain that the whole sequence (x_{z_n}) is weakly convergent in X to x_z. Consequently, the mapping $z \to x_z$ is weakly continuous.

We end this proof with the remark that the mappings $z \to x_z$ and $z \to \theta_z$ are weakly continuous implying that the mapping $z \to \widetilde{x}_z$ is weakly continuous.

Lemma 4.5 *For specified values of k_1 and k_2, the operator Λ has at least one fixed point.*

Proof Let us consider $z = (z_1, z_2) \in \mathcal{K}_1 \times \mathcal{K}_2$, i.e.,

$$\|z_1\|_{L^2(\Gamma_C)} \leqslant k_1, \quad \|z_2\|_{L^2(\Gamma_C)} \leqslant k_2.$$

Then,

$$\|z\|_Y \leqslant k_1 + k_2.$$

Since $z_1 = \psi(u_{z_v} - g)\,\phi_L(\varphi_z - \varphi_{\mathrm{F}})$ and $z_2 = k_{\mathrm{c}}(u_{z_v} - g)\,\phi_L(\theta_z - \theta_{\mathrm{F}})$, it follows from the bounds $|\psi(u_{zv} - g)| \leqslant M_\psi$, $|k_{\mathrm{c}}(u_{zv} - g)| \leqslant M_{k_{\mathrm{c}}}$, and $|\phi_L(\zeta_z - \zeta_{\mathrm{F}})| \leqslant L$ that

$$\|z_1\|_{L^2(\Gamma_C)} \leqslant M_\psi\, L \,\mathrm{meas}(\Gamma_C)^{\frac{1}{2}}, \tag{74}$$

and

$$\|z_2\|_{L^2(\Gamma_C)} \leqslant M_{k_{\mathrm{c}}}\, L \,\mathrm{meas}(\Gamma_C)^{\frac{1}{2}}. \tag{75}$$

From the definition of the operator Λ, we have

$$\|\Lambda z\| \leqslant \|\psi(u_{z_v})\phi_L(\varphi - \varphi_{\mathrm{F}})\| + \|k_{\mathrm{c}}(u_{z_v})\phi_L(\theta - \theta_{\mathrm{F}})\|.$$

Hence, if we select $k_1 = M_\psi L \mathrm{meas}(\Gamma_C)^{\frac{1}{2}}$ and $k_2 = M_{k_{\mathrm{c}}}\ L\ \mathrm{meas}(\Gamma_C)^{\frac{1}{2}}$, we obtain

$$\|\Lambda z\| \leqslant k_1 + k_2.$$

Thus, Λ is an operator from the non-empty, convex, and closed subset $\mathcal{K}_1 \times \mathcal{K}_2$ of $L^2(\Gamma_C)^2$ into itself. Since the space $L^2(\Gamma_C)^2$ is reflexive, $\mathcal{K}_1 \times \mathcal{K}_2$ is weakly compact. The assumptions (h_3)–(h_4) and the continuity of the operators ϕ_L and k_{c}, combined with Lemma 4.4, lead to the weak continuity of Λ. Hence, by Schauder's fixed point theorem, the operator Λ has at least one fixed point.

Now, we have all the ingredients to provide the proof of Theorem 3.1.

(i) Let z^* be the fixed point of the operator Λ and denote by $x^* = (u^*, \varphi^*, \theta^*)$ the solution of the variational problem (PV_z) for $z = z^*$. The definition of Λ and Problem (PV_z) proves that x^* is a solution of Problem (PV), which leads to the existence part of Theorem 3.1.

(ii) We introduce the operators $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{B} : \mathcal{X} \to \mathcal{X}$ defined by

$$(\mathcal{A}\widetilde{x}, \widetilde{y})_{\mathcal{X}} = (Ax, y)_X + (\mathcal{K}\nabla\theta, \nabla\eta)_H, \quad \forall\, \widetilde{x} = (x, \theta), \widetilde{y} = (y, \eta) \in \mathcal{X}, \tag{76}$$

$$(\mathcal{B}\widetilde{x}, \widetilde{y})_{\mathcal{X}} = -(\mathcal{M}\theta, \varepsilon(v))_H - (\mathcal{P}\theta, \nabla\xi)_H, \quad \forall \widetilde{x} = (u, \varphi, \theta),\, \widetilde{y} = (v, \xi, \eta) \in \mathcal{X}, \tag{77}$$

where A is given by (47). We also introduce the functions $\widetilde{j}$, $\widetilde{\ell}$, and $\widetilde{\chi}$ on $\mathcal{X}$ and the element $\widetilde{f} \in \mathcal{X}$ by the following equalities:

$$\widetilde{j}(\widetilde{y}) = \int_{\Gamma_C} S\ \|v_\tau\| \mathrm{d}a, \quad \forall\, \widetilde{y} = (v, \xi, \eta) \in \mathcal{X}, \tag{78}$$

$$\widetilde{\ell}(\widetilde{x}, \widetilde{y}) = \int_{\Gamma_C} \psi(u_\nu - g)\phi_L(\varphi - \varphi_{\mathrm{F}})\xi \mathrm{d}a, \quad \forall\, \widetilde{x} = (u, \varphi, \theta),\, \widetilde{y} = (v, \xi, \eta) \in \mathcal{X}, \tag{79}$$

$$\widetilde{\chi}(\widetilde{x}, \widetilde{y}) = \int_{\Gamma_C} k_{\mathrm{c}}(u_\nu - g)\phi_L(\theta - \theta_{\mathrm{F}})\eta \mathrm{d}a, \quad \forall\, \widetilde{x} = (u, \varphi, \theta),\, \widetilde{y} = (v, \xi, \eta) \in \mathcal{X}, \tag{80}$$

$$\widetilde{f} = (f, q_{\mathrm{e}}, q_{\mathrm{th}}) \in \mathcal{X}. \tag{81}$$

Using (76), (77), (78), (79), (80), (81), and (56), it is easy to see that $\widetilde{x} = (u, \varphi, \theta)$ is a solution of Problem (PV) if and only if

$$\begin{aligned} &(\mathcal{A}\widetilde{x}, \widetilde{y} - \mathcal{X})_{\mathcal{X}} + (\mathcal{B}\widetilde{x}, \widetilde{y} - \widetilde{x})_{\mathcal{X}} + J(\widetilde{y}) - J(\widetilde{x}) + \widetilde{\ell}(\widetilde{x}, \widetilde{y} - \widetilde{x}) + \widetilde{\chi}(\widetilde{x}, \widetilde{y} - \widetilde{x}) \\ &\geqslant (\widetilde{f}, \widetilde{y} - \widetilde{x})_{\mathcal{X}}, \quad \forall\, \widetilde{y} = (v, \xi, \eta) \in \mathcal{X}. \end{aligned} \tag{82}$$

Now, let $\widetilde{x}_1=(u_1,\varphi_1,\theta_1)$ and $\widetilde{x}_2=(u_2,\varphi_2,\theta_2)$ be the solutions of (82). Then,

$$\begin{aligned}&(\mathcal{A}\widetilde{x}_1,\widetilde{y}-\widetilde{x}_1)_{\mathcal{X}}+(\mathcal{B}\widetilde{x}_1,\widetilde{y}-\widetilde{x}_1)_{\mathcal{X}}+J(\widetilde{y})-J(\widetilde{x}_1)+\widetilde{\ell}(\widetilde{x}_1,\widetilde{y}-\widetilde{x}_1)+\widetilde{\chi}(\widetilde{x}_1,\widetilde{y}-\widetilde{x}_1)\\ \geqslant&(\widetilde{f},\widetilde{y}-\widetilde{x}_1)_{\mathcal{X}},\\ &(\mathcal{A}\widetilde{x}_2,\widetilde{y}-\widetilde{x}_2)_{\mathcal{X}}+(\mathcal{B}\widetilde{x}_2,\widetilde{y}-\widetilde{x}_2)_{\mathcal{X}}+J(\widetilde{y})-J(\widetilde{x}_2)+\widetilde{\ell}(\widetilde{x}_2,\widetilde{y}-\widetilde{x}_2)+\widetilde{\chi}(\widetilde{x}_2,\widetilde{y}-\widetilde{x}_2)\\ \geqslant&(\widetilde{f},\widetilde{y}-\widetilde{x}_2)_{\mathcal{X}}.\end{aligned}$$

Take $\widetilde{y}=\widetilde{x}_2$ in the first inequality, $\widetilde{y}=\widetilde{x}_1$ in the second one, and add the two inequalities to get

$$(\mathcal{A}\widetilde{x}_1-\mathcal{A}\widetilde{x}_2,\widetilde{x}_1-\widetilde{x}_2)_{\mathcal{X}}\leqslant G_1+G_2-(\mathcal{B}\widetilde{x}_1-\mathcal{B}\widetilde{x}_2,\widetilde{x}_1-\widetilde{x}_2)_{\mathcal{X}} \tag{83}$$

such that

$$\begin{aligned}G_1&=\widetilde{\ell}(\widetilde{x}_1,\widetilde{x}_2-\widetilde{x}_1)-\widetilde{\ell}(\widetilde{x}_2,\widetilde{x}_1-\widetilde{x}_2),\\ G_2&=\widetilde{\chi}(\widetilde{x}_1,\widetilde{x}_2-\widetilde{x}_1)-\widetilde{\chi}(\widetilde{x}_2,\widetilde{x}_1-\widetilde{x}_2).\end{aligned}$$

By (79) and (80), we have

$$\begin{aligned}G_1=&\int_{\Gamma_C}\psi(u_{2\tau})(\phi_L(\varphi_2-\varphi_{\mathrm{F}})-\phi_L(\varphi_1-\varphi_{\mathrm{F}}))(\varphi_1-\varphi_2)\mathrm{d}a\\ &+\int_{\Gamma_C}\phi_L(\varphi_2-\varphi_{\mathrm{F}})(\psi(u_{2\tau})-\psi(u_{1\tau}))(\varphi_1-\varphi_2)\mathrm{d}a\end{aligned}$$

and

$$\begin{aligned}G_2=&\int_{\Gamma_C}k_{\mathrm{c}}(u_{2\upsilon}-g)(\phi_L(\theta_2-\theta_{\mathrm{F}})-\phi_L(\theta_1-\theta_{\mathrm{F}}))(\theta_1-\theta_2)\mathrm{d}a\\ &+\int_{\Gamma_C}\phi_L(\theta_2-\theta_{\mathrm{F}})(k_{\mathrm{c}}(u_{2\upsilon}-g)-k_{\mathrm{c}}(u_{1\upsilon}-g))(\theta_1-\theta_2)\mathrm{d}a.\end{aligned}$$

Using the proprieties of $\phi_L,\ \psi$, and k_{c}, we deduce

$$\begin{cases}G_1\leqslant(M_\psi c_1^2+L\,L_\psi c_0c_1)\|\widetilde{x}_1-\widetilde{x}_2\|_{\mathcal{X}}^2,\\ G_2\leqslant(M_{k_{\mathrm{c}}}c_2^2+L\,L_{k_{\mathrm{c}}}c_0c_2)\|\widetilde{x}_1-\widetilde{x}_2\|_{\mathcal{X}}^2.\end{cases} \tag{84}$$

Moreover, it follows from (77) and (h_2) that

$$\begin{aligned}&|(\mathcal{B}\widetilde{x}_1-\mathcal{B}\widetilde{x}_2,\widetilde{x}_1-\widetilde{x}_2)|_{\mathcal{X}}\\ \leqslant&\, m^*\|\theta_1-\theta_2\|_{L^2(\Omega)}\|\varepsilon(u_1)-\varepsilon(u_1)\|_H+p^*\|\theta_1-\theta_2\|_{L^2(\Omega)}\|\nabla\varphi_1-\nabla\varphi_2\|_H\\ \leqslant&\max(m^*,p^*)\ \|\theta_1-\theta_2\|_{H^1(\Omega)}\big(\|u_1-u_2\|_V+\|\varphi_1-\varphi_2\|_W\big)\\ \leqslant&\max(m^*,p^*)\ \|\theta_1-\theta_2\|_Q\big\|x_1-x_2\|_X.\end{aligned}$$

Using (56), we find

$$|(\mathcal{B}\widetilde{x}_1 - \mathcal{B}\widetilde{x}_2, \widetilde{x}_1 - \widetilde{x}_2)|_{\mathcal{X}} \leqslant \max\left(\frac{m^*}{2}, \frac{p^*}{2}\right) \|\widetilde{x}_1 - \widetilde{x}_2\|^2_{\mathcal{X}}. \tag{85}$$

Using the properties of truncation operators A and $\mathcal{K}$, it is easy to see that the operator $\mathcal{A}$ is a strongly monotone Lipschitz continuous operator on $\mathcal{X}$.

Finally, we combine (83) and (84)–(85) to prove that there exists a constant $c^* > 0$ such that

$$\|\widetilde{x}_1 - \widetilde{x}_2\|^2_{\mathcal{X}} \leqslant c^*(M_\psi + M_{k_c} + LL_\psi + LL_{k_c} + \max(m^*, p^*))\|\widetilde{x}_1 - \widetilde{x}_2\|^2_{\mathcal{X}}.$$

Let $L^* = \frac{1}{c^*}$. If

$$M_\psi + M_{k_c} + LL_\psi + LL_{k_c} + \max(m^*, p^*) < L^*,$$

then we obtain $\widetilde{x}_1 = \widetilde{x}_2$, which ensures the uniqueness.

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